1. (Product manifold). Let M and N be differentiable manifolds and let $\{(U_\alpha, x_\alpha)\}$, $\{(V_\beta, y_\beta)\}$ differentiable structures on M and N, respectively. Consider the Cartesian product $M\times N$ and the mappings $z_{\alpha\beta}(p,q)=(x_\alpha(p), y_\beta(q))$, $p\in U_\alpha$, $q\in V_\beta$.

(a) Prove that $\{(U_\alpha\times V_\beta, z_{\alpha\beta})\}$ $\boxed{\text{is}}$ (determines?) a differentiable structure on $M\times N$ in which the projections $\pi_1: M\times N \longrightarrow M$ and $\pi_2: M\times N\longrightarrow N$ are differentiable. With this differentiable structure $M\times N$ is called the $\boxed{\text{product manifold}}$ of M with N.

(b) Show that the product manifold $S^1\times\cdots\times S^1$ of n circles S^1, where $S^1\subset\mathbb{R}^2$ has the usual differentiable structure, is diffeomorphic to the n-torus T^n of example 4.9 (a).

• Proof: (a) If $z_{\alpha_1\beta_1}(U_{\alpha_1}\times V_{\beta_1})\cap z_{\alpha_2\beta_2}(U_{\alpha_2}\times V_{\beta_2})\neq\phi$, then $x_{\alpha_1}(U_{\alpha_1})\cap x_{\alpha_2}(U_{\alpha_2})\neq\phi$ and $y_{\beta_1}(V_{\beta_1})\cap y_{\beta_2}(V_{\beta_2})\neq\phi$.

Assume $(p,q)\in z^{-1}_{\alpha_1\beta_1}\left(z_{\alpha_1\beta_1}(U_{\alpha_1}\times V_{\beta_1})\cap z_{\alpha_2\beta_2}(U_{\alpha_2}\times V_{\beta_2})\right)$, then

$$\left(z^{-1}_{\alpha_2\beta_2}\circ z_{\alpha_1\beta_1}\right)(p,q) = z^{-1}_{\alpha_2\beta_2}\left(x_{\alpha_1}(p), y_{\beta_1}(q)\right) = \left(x^{-1}_{\alpha_2}\circ x_{\alpha_1}(p),\ y^{-1}_{\beta_2}\circ y_{\beta_1}(q)\right)\in\mathbb{R}^{\dim M+\dim N}$$

which is differentiable.

Moreover, it is direct to see that $\bigcup_{\alpha,\beta} z_{\alpha\beta}(U_\alpha\times V_\beta)\supset M\times N$.

Finally, it is not necessarily maximal.

So $\{(U_\alpha\times V_\beta, z_{\alpha\beta})\}$ determines a differentiable structure on $M\times N$.

$\pi_1: M\times N\to M$ is differentiable because $(x_\alpha^{-1}\circ\pi_1\circ z_{\alpha\beta})(x_1,\dots,x_m,y_1,\dots,y_n)=(x_1,\dots,x_m)$. Similar for π_2.

(b) Denote $S^1\times\cdots\times S^1 = \{(e^{2\pi i\theta_1}, e^{2\pi i\theta_2},\dots,e^{2\pi i\theta_n}) \mid 0\le\theta_j<1,\ j=1,\dots,n\}$.

Let $\pi:\mathbb{R}^n\to T^n$ be the natural projection

$$\pi: (\theta_1,\dots,\theta_n)\longmapsto([\theta_1],\dots,[\theta_n])\in(\mathbb{R} \bmod 1)^n.$$

Define $f: S^1\times\cdots\times S^1\longrightarrow T^n$ by

$$\left(e^{2\pi i\theta_1}, e^{2\pi i\theta_2},\dots,e^{2\pi i\theta_n}\right)\longmapsto \pi(\theta_1,\dots,\theta_n)$$

then f is obviously well-defined, 1-1 and onto. It remains to show that f is a diffeomorphism. Consider the following parametrizations:

$$\mathbb{R}^n \xrightarrow{\pi_N^{-1}} S^1\times\cdots\times S^1\setminus\{(0,1)\times\cdots\times(0,1)\} \xrightarrow{f} U\subset\mathbb{R}^n \longrightarrow \pi(U)\subset T^n$$

$$(x_1,\dots,x_n)\longmapsto\left(\frac{2x_1}{x_1^2+1}+i\frac{x_1^2-1}{x_1^2+1},\ \dots,\ \frac{2x_n}{x_n^2+1}+i\frac{x_n^2-1}{x_n^2+1}\right)\longmapsto\left(\frac{1}{\pi}\arctan\frac{x_1^2-1}{2x_1}+\frac{1}{2},\ \dots,\ \frac{1}{\pi}\arctan\frac{x_n^2-1}{2x_n}+\frac{1}{2}\right)$$

No.

Date

Thus f is differentiable. (If we choose another chart on $S^1 \times \cdots \times S^1$, some coordinates x_j are mapped to

$$x_j \longmapsto \frac{2x_j}{x_j^2+1} - i\,\frac{x_j^2-1}{x_j^2+1} \longmapsto -\frac{1}{\pi}\arctan\frac{x_j^2-1}{2x_j} + \frac{1}{2}$$

which is still differentiable) For f^{-1}, not that when $\theta - 1/2 \in [-1/2, 1/2]$, we always have $\cos\pi(\theta - 1/2) \geq 0$. Thus $(\theta_1, \cdots, \theta_n) \in T^n$ is mapped to

$$\left(\frac{1}{\sqrt{1+\tan^2\pi(\theta_1-1/2)}} + i\,\frac{\tan\pi(\theta_1-1/2)}{\sqrt{1+\tan^2\pi(\theta_1-1/2)}}, \cdots, \frac{1}{\sqrt{1+\tan^2\pi(\theta_n-1/2)}} + i\,\frac{\tan\pi(\theta_n-1/2)}{\sqrt{1+\tan^2\pi(\theta_n-1/2)}}\right) \in S^1_{\times}\cdots{}_{\times}S^1 \setminus N^1$$

in which we denote $N^1 = (0,1) \times \cdots \times (0,1)$. In local coordinate chart it is mapped to

$$\left(\frac{1}{\sqrt{1+\tan^2\pi(\theta_j-1/2)}} \Bigg/ \left(1 - \frac{\tan\pi(\theta_j-1/2)}{\sqrt{1+\tan^2\pi(\theta_j-1/2)}}\right)\right)_{1\leq j\leq n} = \left(\sqrt{1+\tan^2\pi(\theta_j-1/2)} + \tan\pi(\theta_j-1/2)\right)_{1\leq j\leq n}$$

which is also differentiable with respect to each θ_j, $1\leq j\leq n$.

(If we choose another chart on $S^1 \times \cdots \times S^1$, some coordinates θ_j are mapped to

$$\theta_j \longmapsto \frac{1}{\sqrt{1+\tan^2\pi(1/2-\theta_j)}} + i\,\frac{\tan\pi(1/2-\theta_j)}{\sqrt{1+\tan^2\pi(1/2-\theta_j)}} \longmapsto \sqrt{1+\tan^2\pi(\theta_j-1/2)} + \tan\pi(1/2-\theta_j)$$

which is still differentiable in θ_j.)

Thus the product manifold $S^1 \times \cdots \times S^1$ with the product differentiable structure is diffeomorphic to the differentiable manifold $S^1 \times \cdots \times S^1$ with the differentiable structure induced by the properly discontinuous actions of the group of translations.

2. Prove that the tangent bundle of a differentiable manifold M is orientable (even though M may not be).

- Proof: Assume $\{(\mathcal{U}_\alpha, h_\alpha)\}$ is an atlas for M. Then a differentiable structure on TM is given by $\{(\mathcal{U}_\alpha \times \mathbb{R}^n, F_\alpha)\}$, where $F_\alpha : \mathcal{U}_\alpha \times \mathbb{R}^n \longrightarrow TM$ is given by

$$F(x_1, \cdots, x_n, v_1, \cdots, v_n) = \left(h_\alpha(x_1, \cdots, x_n),\ dh_\alpha\left(v_1 \frac{\partial}{\partial x_1} + \cdots + v_n \frac{\partial}{\partial x_n}\right)\right)$$

where $dh_\alpha : T_{(x_1, \cdots, x_n)}\mathbb{R}^n \longrightarrow T_{h_\alpha(x_1, \cdots, x_n)}M$ is the tangent map of h_α at $h_\alpha(x_1, \cdots, x_n) \in M$.

Now let $(\mathcal{U}_\alpha \times \mathbb{R}^n, F_\alpha)$ and $(\mathcal{U}_\beta \times \mathbb{R}^n, F_\beta)$ be two coordinate charts of TM with $\mathcal{U}_\alpha \cap \mathcal{U}_\beta \neq \phi$. Then the transition map from $\mathcal{U}_\alpha \times \mathbb{R}^n$ to $\mathcal{U}_\beta \times \mathbb{R}^n$ is given by

$$(x_1, \cdots, x_n) \longmapsto \left(\left(h_\beta^{-1} \circ h_\alpha\right)(x_1, \cdots, x_n),\ d\left(h_\beta^{-1} \circ h_\alpha\right)\Big|_{(x_1, \cdots, x_n)} (v_1, \cdots, v_n)\right).$$

Hence the Jacobian matrix of $F_\beta^{-1} \circ F_\alpha$ is

$$J = \begin{pmatrix} d(h_\beta^{-1} \circ h_\alpha) & 0 \\ * & d(h_\beta^{-1} \circ h_\alpha) \end{pmatrix}$$

which gives $\det J = \left|\det\left(d(h_\beta^{-1} \circ h_\alpha)\right)\right|^2 > 0$, where $\det\left(d(h_\beta^{-1} \circ h_\alpha)\right) \neq 0$ because $h_\beta^{-1} \circ h_\alpha$ is a diffeomorphism and thus $d(h_\beta^{-1} \circ h_\alpha)$ is invertible.

Therefore we have shown that TM is always orientable.

No.

Date

3. Prove that:

(a) a regular surface $S \subset \mathbb{R}^3$ is an orientable manifold if and only if there exists a differentiable mapping of $N: S \longrightarrow \mathbb{R}^3$ with $N(p) \perp T_p(S)$ and $|N(p)| = 1$, for all $p \in S$.

(b) the Möbius band (Example 4.9 (b)) is non-orientable.

- Proof: (a) "$\Rightarrow$": Pick an arbitrary point $p \in S$, and write $q = x^{-1}(p)$, where x is a local coordinate map from $U \subset \mathbb{R}^2$ onto $S \cap V$, where $V \subset \mathbb{R}^3$ is an open neighborhood around p. Find two linearly independent vectors $v_1, v_2 \in T_q\mathbb{R}^2$. Since $(dx)_q : T_q\mathbb{R}^2 \longrightarrow T_pS$ is injective, $(dx)_q(v_1)$ and $(dx)_q(v_2)$ are also linearly independent. Thus $(dx)_q(v_1) \wedge (dx)_q(v_2) \neq 0$ and we can define

$$N: S \longrightarrow \mathbb{R}^3$$

$$p \longmapsto \frac{(dx)_q(v_1) \wedge (dx)_q(v_2)}{\|(dx)_q(v_1) \wedge (dx)_q(v_2)\|}$$

That the vector $N(p)$ is independent of choices of linearly independent tangent vectors $v_1, v_2 \in T_q\mathbb{R}^2$ is obvious from the geometric interpretation. Since $T_{x^{-1}(p)}\mathbb{R}^2$ are all isomorphic for all $p \in S$, we can fix v_1 and $v_2 \in \mathbb{R}^2$. To show that N is well-defined, let $y: \tilde{U} \longrightarrow S \cap V$ be another chart. Then orientability gives $\det\left(\frac{\partial y_i}{\partial x_j}\right) > 0$. Thus

$$\frac{(dy)_{y^{-1}(p)}(v_1) \wedge (dy)_{y^{-1}(p)}(v_2)}{\|(dy)_{y^{-1}(p)}(v_1) \wedge (dy)_{y^{-1}(p)}(v_2)\|} = \frac{\det\left(\frac{\partial y_i}{\partial x_j}\right)}{\left|\det\left(\frac{\partial y_i}{\partial x_j}\right)\right|} \cdot \frac{(dx)_{x^{-1}(p)}(v_1) \wedge (dx)_{x^{-1}(p)}(v_2)}{\|(dx)_{x^{-1}(p)}(v_1) \wedge (dx)_{x^{-1}(p)}(v_2)\|}$$

$$= 1 \cdot \frac{(dx)_{x^{-1}(p)}(v_1) \wedge (dx)_{x^{-1}(p)}(v_2)}{\|(dx)_{x^{-1}(p)}(v_1) \wedge (dx)_{x^{-1}(p)}(v_2)\|}$$

$$= \frac{(dx)_{x^{-1}(p)}(v_1) \wedge (dx)_{x^{-1}(p)}(v_2)}{\|(dx)_{x^{-1}(p)}(v_1) \wedge (dx)_{x^{-1}(p)}(v_2)\|}$$

which gives the well-definedness of the mapping $N: S \longrightarrow \mathbb{R}^3$.

"$\Leftarrow$": Without loss of generality, assume all charts are connected. Note that at each $p \in S$, $N(p)$ and $\dfrac{(dx)_q(v_1) \wedge (dx)_q(v_2)}{\|(dx)_q(v_1) \wedge (dx)_q(v_2)\|} =: \eta(p)$ are linearly dependent.

Note that $|N(p)| = 1$, $\forall p \in S$.

Thus either $N(p) = \eta(p)$ or $N(p) = -\eta(p)$. We can always choose the coordinate maps such that $N(p) = \eta(p)$. Now we show such an atlas must be oriented. This is because under a change of coordinates

$$\frac{(dy)_{y^{-1}(p)}(v_1) \wedge (dy)_{y^{-1}(p)}(v_2)}{\|(dy)_{y^{-1}(p)}(v_1) \wedge (dy)_{y^{-1}(p)}(v_2)\|} = \frac{\det\left(\frac{\partial y_i}{\partial x_j}\right)}{\left|\det\left(\frac{\partial y_i}{\partial x_j}\right)\right|} \frac{(dx)_{x^{-1}(p)}(v_1) \wedge (dx)_{x^{-1}(p)}(v_2)}{\|(dx)_{x^{-1}(p)}(v_1) \wedge (dx)_{x^{-1}(p)}(v_2)\|}$$

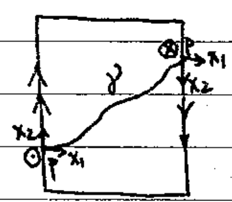
i.e. $N(p) = \dfrac{\det(\partial y_i/\partial x_j)}{|\det(\partial y_i/\partial x_j)|} \cdot N(p) \Rightarrow \det(\partial y_i/\partial x_j) = |\det(\partial y_i/\partial x_j)|$

$\Rightarrow \det(\partial y_i/\partial x_j) > 0$

$\Rightarrow$ this atlas is orientable

$\Rightarrow$ S is orientable

(b) Such a normal vector field does not exist on a Möbius band.

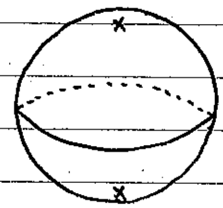


To make this precise, let γ be an arbitrary connected path on the Möbius band which is the generator of $H_1(\text{Möbius})$. By observation on the orientation of the frame, we see there should be different pointing directions of the normal vector at the same point (this follows from the construction of $N: S \to \mathbb{R}^3$ given in (a)). Thus $N(p) = 0$, contradicting "$|N(p)| = 1$ for all $p \in S$".

4. Show that the projective plane $\mathbb{RP}^2$ is non-orientable.

Hint: Prove that if the manifold M is orientable, then any open subset of M is an orientable submanifold. Observe that $\mathbb{RP}^2$ contains an open subset diffeomorphic to a Möbius band, which is non-orientable.

- Proof: The restriction does not change the Jacobian (because it depends only on the local data). Thus if M is orientable then any open subset of M is an orientable submanifold.
- Why do we need the open subset of $\mathbb{RP}^2$ "diffeomorphic" to a Möbius band? This open subset is obviously homeomorphic to a Möbius band. And it has a differentiable structure inherited from $\mathbb{RP}^2$. If we know the uniqueness of the differentiable structure on the Möbius band, then it has to be the same and we get the required "diffeomorphic" property.
- 从另一个角度来看，orientability 应是某种拓扑概念，因此强求 diffeomorphic 是有点 demanding a little bit too much 了… homeomorphic 就好。如果这样的话就已经证完了…
- 如果非要证 diffeomorphic 怎么办？好吧，根据 example 4.9(b) 的精神，$\pi: S^3 \to \mathbb{RP}^2$ 是个 local diffeomorphism，而 π: Cylinder $\longrightarrow$ Möbius band 也是个 local diffeomorphism. 所以只要 $S^3 \setminus \{N, S\}$ 和 Cylinder 是 diffeomorphic 的就行了。

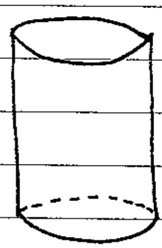


$(\cos\theta\cos\varphi, \cos\theta\sin\varphi, \sin\theta)$

$-\pi < \theta < \pi, \quad 0 \le \varphi \le 2\pi$

$x^2+y^2+z^2=1, \ z \neq \pm 1$

$$(x, y, z) \xmapsto{\text{diffeomorphism 达成}} \left(\frac{x}{\sqrt{1-z^2}}, \frac{y}{\sqrt{1-z^2}}, z\right)$$



$(\cos\varphi, \sin\varphi, h)$

$0 \le \varphi \le 2\pi, \ -1 < h \le 1$

5. (Embedding of $\mathbb{RP}^2$ in $\mathbb{R}^4$). Let $F : \mathbb{R}^3 \longrightarrow \mathbb{R}^4$ be given by

$$F(x,y,z) = (x^2-y^2,\ xy,\ xz,\ yz), \qquad (x,y,z) = p \in \mathbb{R}^3.$$

Let $S^2 \subset \mathbb{R}^3$ be the unit sphere with the origin $0 \in \mathbb{R}^3$. Observe that the restriction $\varphi = F|S^2$ is such that $\varphi(p) = \varphi(-p)$, and consider the mapping $\tilde{\varphi} : \mathbb{RP}^2 \longrightarrow \mathbb{R}^4$ given by $\tilde{\varphi}([p]) = \varphi(p)$, $[p]$ = equiv. class of $p = \{p, -p\}$.

Prove that:

(a) $\tilde{\varphi}$ is an immersion.

(b) $\tilde{\varphi}$ is injective; together with (a) and the compactness of $\mathbb{RP}^2$, this implies that $\tilde{\varphi}$ is an embedding.

- Proof: (a)

$$\begin{tikzcd} S^2 \arrow[r, hook, "i"] \arrow[d, "\pi"'] & \mathbb{R}^3 \arrow[r, "F"] & \mathbb{R}^4 \\ \mathbb{RP}^2 \arrow[urr, "\tilde{\varphi}"'] & & \end{tikzcd}$$

$F \circ i = \varphi.$

$\varphi = \tilde{\varphi} \circ \pi.$

First, $\tilde{\varphi}$ is a smooth map from $\mathbb{RP}^2$ to $\mathbb{R}^4$. Take for example the chart on $\mathbb{RP}^2$ $V_z := \{[x,y,z] \in \mathbb{RP}^2 : z \neq 0\}$. Let $\psi_z : \mathbb{R}^2 \longrightarrow V_z$ denote the coordinate map $\mathbb{R}^2 \ni (x,y) \longmapsto [x,y,1] \in V_z \subset \mathbb{RP}^2$. Then at any $p = \psi_z((x,y)) \in \mathbb{RP}^2$, we have

$$\left[(id)^{-1} \circ \tilde{\varphi} \circ \psi_z\right]((x,y)) = \tilde{\varphi}([x,y,1]) = \varphi\left(\left(\frac{x}{\sqrt{x^2+y^2+1}}, \frac{y}{\sqrt{x^2+y^2+1}}, \frac{1}{\sqrt{x^2+y^2+1}}\right)\right)$$

$$= \left(\frac{x^2-y^2}{x^2+y^2+1}, \frac{xy}{x^2+y^2+1}, \frac{x}{x^2+y^2+1}, \frac{y}{x^2+y^2+1}\right) \in \mathbb{R}^4$$

which is smooth. Similar for other charts on $\mathbb{RP}^2$.

Second, $\tilde{\varphi}$ is an immersion. Note that under the above coordinate charts,

$$\underset{(\text{on } V_z)}{(d\tilde{\varphi})_p} = \begin{pmatrix} \frac{2x(x^2+y^2+1) - (x^2-y^2)\cdot 2x}{(x^2+y^2+1)^2} & \frac{y(x^2+y^2+1) - xy\cdot 2x}{(x^2+y^2+1)^2} & \frac{1\cdot(x^2+y^2+1) - x\cdot 2x}{(x^2+y^2+1)^2} & \frac{0 - y\cdot 2x}{(x^2+y^2+1)^2} \\ \frac{-2y(x^2+y^2+1) - (x^2-y^2)\cdot 2y}{(x^2+y^2+1)^2} & \frac{x(x^2+y^2+1) - xy\cdot 2y}{(x^2+y^2+1)^2} & \frac{0 - x\cdot 2y}{(x^2+y^2+1)^2} & \frac{1\cdot(x^2+y^2+1) - y\cdot 2y}{(x^2+y^2+1)^2} \end{pmatrix}$$

$$= \frac{1}{(x^2+y^2+1)^2} \begin{pmatrix} 2x(2y^2+1) & y(-x^2+y^2+1) & -x^2+y^2+1 & -2xy \\ -2y(2x^2+1) & x(x^2-y^2+1) & -2xy & x^2-y^2+1 \end{pmatrix}$$

No.

Date

We want to show that $(d\tilde{\varphi})_p$ is of rank 2. Note that

$$\det\begin{pmatrix} -x^2+y^2+1 & -2xy \\ -2xy & x^2-y^2+1 \end{pmatrix} = \left[-(x^2-y^2)+1\right]\left[(x^2-y^2)+1\right] - 4x^2y^2$$
$$= 1-(x^2-y^2)^2-4x^2y^2 = 1-(x^2+y^2)^2$$

$$\det\begin{pmatrix} y(-x^2+y^2+1) & -x^2+y^2+1 \\ x(x^2-y^2+1) & -2xy \end{pmatrix} = -2xy^2(-x^2+y^2+1) - x(x^2-y^2+1)(-x^2+y^2+1) = -x(-x^2+y^2+1)(x^2+y^2+1)$$

$$\det\begin{pmatrix} y(-x^2+y^2+1) & -2xy \\ x(x^2-y^2+1) & x^2-y^2+1 \end{pmatrix} = y(-x^2+y^2+1)(x^2-y^2+1)+2x^2y(x^2-y^2+1) = y(x^2+y^2+1)(x^2-y^2+1).$$

If all three determinants above are zero, then

$$\begin{cases} x^2+y^2=1 \\ -x(y^2-1+y^2+1) = -2xy^2=0 \\ (1-y^2-y^2+1)y = -y^3=0 \end{cases} \Rightarrow y=0,\ x^2=1.$$

But in this case

$$\det\begin{pmatrix} 2x(2y^2+1) & y(-x^2+y^2+1) \\ -2y(2x^2+1) & x(x^2-y^2+1) \end{pmatrix} = \det\begin{pmatrix} \pm 2 & 0 \\ 0 & \pm 2 \end{pmatrix} = 4 \neq 0.$$

Thus $\operatorname{rank}(d\tilde{\varphi})_p = 2$. Similarly we can show that $\operatorname{rank}(d\tilde{\varphi})_p=2$ for all $p\in\mathbb{RP}^2$ (not just $p\in V_z$). Therefore $\tilde{\varphi}$ is an immersion.

(b) To verify that $\tilde{\varphi}$ is an injective, we only need to verify that $\varphi\circ i\circ\pi^{-1}$ is injective. Assume $\varphi(P_1)=\varphi(P_2)$, $P_1,P_2\in S^2$, $P_1=(x_1,y_1,z_1)$, $P_2=(x_2,y_2,z_2)$, then it follows that

$$\begin{cases} x_1^2-y_1^2=x_2^2-y_2^2 \\ x_1y_1=x_2y_2 \\ x_1z_1=x_2z_2 \\ y_1z_1=y_2z_2 \end{cases} \Rightarrow \begin{cases} x_1^2=x_2^2 \\ y_1^2=y_2^2 \\ z_1^2=z_2^2 \end{cases}$$

Since $x_1^2+y_1^2+z_1^2=1=x_2^2+y_2^2+z_2^2$, x_1,y_1,z_1 can't vanish simultaneously. Neither can x_2,y_2,z_2. There're two cases to consider.

(i) If there are two non-vanishing coordinates in P_1 (thus also in P_2), w.l.o.g. assume $x_1 \neq 0$ and $y_1 \neq 0$, then by $x_1^2 = x_2^2$ and $y_1^2 = y_2^2$ we know $x_2 \neq 0$ and $y_2 \neq 0$. And it follows from $x_1 y_1 = x_2 y_2$ that, either $x_1 = x_2, y_1 = y_2$ or $x_1 = -x_2$, $y_1 = -y_2$. In the first case, by $y_1 z_1 = y_2 z_2$ we have $z_1 = z_2$ and thus $(x_1, y_1, z_1) = (x_2, y_2, z_2)$; in the second case, by $y_1 z_1 = y_2 z_2$ we have $z_1 = -z_2$ and thus $(x_1, y_1, z_1) = -(x_2, y_2, z_2)$.

(ii) If there is only one non-vanishing coordinate in P_1 (thus also in P_2), w.l.o.g. assume $x_1 \neq 0$ (and hence $x_2 \neq 0$). Then by $x_1^2 = x_2^2$ we have $x_1 = \pm x_2$, and hence $P_1 = (x_1, 0, 0) = \pm(x_2, 0, 0) = \pm P_2$.

In either Case (i) or Case (ii), we will have $\pi(P_1) = \pi(P_2) = [p]$. i.e. if $\tilde{\varphi}([p]) = \tilde{\varphi}([q])$ then $[p] = [q]$. Thus $\tilde{\varphi}$ is injective.

Finally, in order to show that $\tilde{\varphi}$ is an embedding, we only need to show that $\tilde{\varphi}$ is a homeomorphism from $\mathbb{RP}^2$ onto $\tilde{\varphi}(\mathbb{RP}^2) \subset \mathbb{R}^4$ under the subspace topology induced from $\mathbb{R}^4$.

- We can cite the fact that "a bijective continuous map from a compact space to a Hausdorff space is a homeomorphism between its domain and image".

- To see a direct approach: note that $\mathbb{RP}^2$ is compact, and $\tilde{\varphi}$ is a bijection from $\mathbb{RP}^2$ to $\tilde{\varphi}(\mathbb{RP}^2)$. On the one hand, $\tilde{\varphi}$ is continuous by its construction; on the other hand, for any closed subset $A \subset \mathbb{RP}^2$, A is compact since $\mathbb{RP}^2$ is compact, and thus $\tilde{\varphi}(A)$ is compact in $\mathbb{R}^4$ under the subspace topology due to the continuity of $\tilde{\varphi}$. Moreover, $\tilde{\varphi}(A)$ is indeed closed in $\mathbb{R}^4$ under the subspace topology because it is compact in a Hausdorff space ($\mathbb{R}^4$). Hence the preimage of any closed subset of $\mathbb{RP}^2$ under the map $\tilde{\varphi}^{-1}$ (which exists because $\tilde{\varphi}$ is bijective) is closed in $\tilde{\varphi}(\mathbb{RP}^2)$ under the subspace topology induced from $\mathbb{R}^4$. In other words, $\tilde{\varphi}^{-1}$ is continuous. Therefore $\tilde{\varphi}$ is a homeomorphism.

Together with (a) we know $\tilde{\varphi}: \mathbb{RP}^2 \longrightarrow \mathbb{R}^4$ is an embedding by definition.

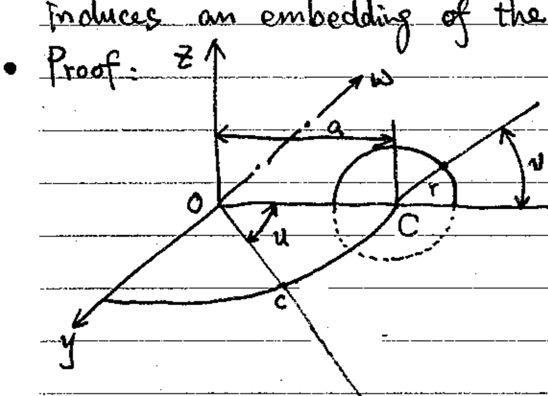
1. For a smooth manifold M^n, TM is an orientable smooth manifold of dimension $2n$. Does there exist a complex structure on TM?

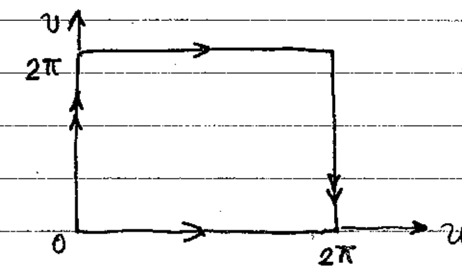
2. What is the obstruction for the existence of a diffeomorphism between two smooth manifolds?

Date

6. (Embedding of the Klein bottle in $\mathbb{R}^4$). Show that the mapping $G: \mathbb{R}^2 \to \mathbb{R}^4$ given by $G(x,y) = \left((r\cos y + a)\cos x, (r\cos y + a)\sin x, r\sin y \cos\frac{x}{2}, r\sin y \sin\frac{x}{2}\right)$, $(x,y) \in \mathbb{R}^2$ induces an embedding of the Klein bottle (Example 4.9 (b)) into $\mathbb{R}^4$.

• Proof:





Let $U_1 = \{(x,y) \in \mathbb{R}^2 : 0 < x < 2\pi, 0 < y < 2\pi\}$ be a coordinate chart in $\mathbb{R}^2$. The coordinate map given by $\varphi: U_1 \subset \mathbb{R}^2 \to K$. Then

$$(G\circ\varphi)(x,y) = \left((r\cos y + a)\cos x, (r\cos y + a)\sin x, r\sin y\cos\frac{x}{2}, r\sin y \sin\frac{x}{2}\right) \in \mathbb{R}^4.$$

and thus

$$d(G\circ\varphi)\Big|_{(x,y)} = \begin{pmatrix} -(r\cos y + a)\sin x & (r\cos y + a)\cos x & -\frac{r}{2}\sin y \sin\frac{x}{2} & \frac{r}{2}\sin y\cos\frac{x}{2} \\ (-r\sin y + a)\cos x & (-r\sin y + a)\sin x & r\cos y\cos\frac{x}{2} & r\cos y \sin\frac{x}{2} \end{pmatrix}$$

Note that

$$d(G\circ\varphi)\Big|_{(x,y)} \cdot \left(d(G\circ\varphi)\Big|_{(x,y)}\right)^T = \begin{pmatrix} (r\cos y + a)^2 + \frac{r^2}{4}\sin^2 y & 0 \\ 0 & (-r\sin y + a)^2 + r^2\cos^2 y \end{pmatrix}$$

Which is nonsingular. (since otherwise either $\begin{cases} r\cos y = -a \\ r\sin y = 0 \end{cases}$ or $\begin{cases} r\sin y = a \\ r\cos y = 0 \end{cases}$

both cases contradict $a > r > 0$.)

Hence $d(G\circ\varphi)\Big|_{(x,y)}$ is of full rank.

i.e. $\operatorname{rank} d(G\circ\varphi)\Big|_{(x,y)} = 2$.

By definition it follows that G is an immersion from $K \supset \varphi(U_1)$ to $\mathbb{R}^4$. Similarly we can show results for U_2 and U_3, concluding that G is an immersion from K to $\mathbb{R}^4$.

Next, we want to show that G is injective. To see that, let us first assume that $r\sin y\cos\frac{x}{2} \neq 0$. Then both $\sin y$ and $\cos\frac{x}{2}$ are nonzero.

Since $0 < x/2 < \pi$, the expression $\frac{r\sin y \sin\frac{x}{2}}{r\sin y \cos\frac{x}{2}} = \tan\frac{x}{2}$ uniquely determines x.
Then we can use that

$$\sin y = \frac{r\sin y \sin\frac{x}{2}}{r\sin\frac{x}{2}}, \quad \cos y = \frac{+\sqrt{(r\cos y+a)^2\cos^2 x + (r\cos y + a)^2 \sin^2 x} - a}{r}$$

to uniquely determine y. This proves the injectivity when $r\sin y\cos\frac{x}{2} \neq 0$.
If $r\sin y\cos\frac{x}{2} = 0$, then $y = \pi$ or $x = \pi$, and again injectivity is easily checked.
Similarly we can show results for U_2 and U_3, concluding that G is injective from K to $\mathbb{R}^4$.
Finally, note that K is compact, $\mathbb{R}^4$ is Hausdorff, it follows from the same pattern of argument as in Problem 5 that G induces an embedding of the Klein bottle into $\mathbb{R}^4$.

No.

Date

7. (Infinite Möbius band). Let $C = \{(x,y,z) \in \mathbb{R}^3;\ x^2+y^2=1\}$ be a right circular cylinder, and let $A: C \longrightarrow C$ be the symmetry with respect to the origin $O \in \mathbb{R}^3$, that is, $A(x,y,z) = (-x,-y,-z)$. Let M be the quotient space of C with respect to the equivalence relation $p \sim A(p)$, and let $\pi: C \longrightarrow M$ be the projection $\pi(p) = \{p, A(p)\}$.

(a) Show that it is possible to give M a differentiable structure such that π is a local diffeomorphism.

(b) Prove that M is non-orientable.

• Proof: (a) Note that C is a differentiable manifold, and the group $G=\{Id, A\}$ acts properly discontinuously on C. Thus it follows from the argument on pp. 23 that $M = C/G$ has a differentiable structure with respect to which the projection $\pi: C \to M = C/G$ is a local diffeomorphism.

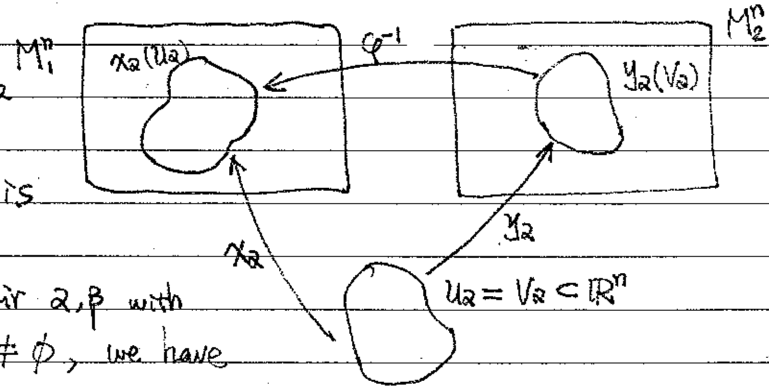
(b) M contains an open submanifold $M' = C'/G = \{(x,y,z) \in \mathbb{R}^3: x^2+y^2=1, -1<z<1\}/G$, which is non-orientable as shown in Problem 3(b). Then apply the hint for Problem 4 to conclude that $M = C/G$ is non-orientable.

8. Let M_1 and M_2 be differentiable manifolds. Let $\varphi: M_1 \longrightarrow M_2$ be a local diffeomorphism. Prove that if M_2 is orientable, then M_1 is orientable.

- Proof: Since M_2 is orientable, there exists an atlas $\{(\tilde{V}_\alpha, \tilde{y}_\alpha)\}$ of M_2 such that for every pair α, β with $\tilde{y}_\alpha(\tilde{V}_\alpha) \cap \tilde{y}_\beta(\tilde{V}_\beta) \neq \phi$, the differential of the change of coordinates $y_\beta^{-1} \circ y_\alpha$ has positive determinant. It is easy to verify that any refinement of this atlas (with coordinate maps induced from restrictions of those $\tilde{y}_\alpha$'s) is again an oriented atlas. Now that $\varphi: M_1 \longrightarrow M_2$ is a local diffeomorphism, for any $p \in M_1$ there exists neighborhoods U of p and V of $\varphi(p)$ such that $\varphi: U \longrightarrow V$ is a diffeomorphism. We can choose each V small enough such that V is always contained in some $\tilde{V}_\alpha$. Note that $\varphi(M_1)$ is an open submanifold of M_2, thus $\varphi(M_1)$ is orientable and $\{(\tilde{V}_\alpha, \tilde{y}_\alpha)\}$ restricted to $\varphi(M_1)$ is an oriented atlas. The refinement constructed above by φ then is an oriented atlas on $\varphi(M_1)$. Denote this atlas by $\{(V_\alpha, y_\alpha)\}$. Construct an atlas on M_1 (denoted by $\{(\mathcal{U}_\alpha, x_\alpha)\}$) by setting

$$\mathcal{U}_\alpha = V_\alpha, \qquad x_\alpha = \varphi^{-1} \circ y_\alpha : \mathcal{U}_\alpha \longrightarrow M_1$$

It is easy to verify that $\{(\mathcal{U}_\alpha, x_\alpha)\}$ is a differentiable structure on M_1. We are left to show that it is oriented. To see this, note that for every pair α, β with $x_\alpha(\mathcal{U}_\alpha) \cap x_\beta(\mathcal{U}_\beta) = W \neq \phi$, we have



$$\det\left(d\left(x_\beta^{-1} \circ x_\alpha\right)\right) = \det\left(d\left(y_\beta^{-1} \circ \varphi \circ \varphi^{-1} \circ y_\alpha\right)\right) = \det\left(d\left(y_\beta^{-1} \circ y_\alpha\right)\right) > 0$$

at each point in $\mathcal{U}_\alpha = V_\alpha$. Thus $\{(\mathcal{U}_\alpha, x_\alpha)\}$ is an oriented atlas and hence M_1 is orientable.

Remark: Another proof can be given by pulling back a globally nowhere vanishing n-form on M_2^n to a globally nowhere vanishing n-form on M_1^n by means of the local diffeomorphism φ. (Start with the observation that $\dim M_1 = \dim M_2$ by the existence of the local diffeomorphism φ, and assume without loss of generality that both M_1 and M_2 are connected.)

No.

Date

9. Let $G \times M \longrightarrow M$ be a properly discontinuous action of a group G on a differentiable manifold M.

(a) Prove that the manifold M/G (Example 4.8) is orientable if and only if there exists an orientation of M that is preserved by all the diffeomorphisms of G.

(b) Use (a) to show that the projective plane $\mathbb{RP}^2$, the Klein bottle and the Möbius band are non-orientable.

(c) Prove that $\mathbb{RP}^n$ is orientable if and only if n is odd.

• Proof: (a) "$\Rightarrow$": Assume there exists an orientation of M (denoted as $\{(f_a, U_a, V_a)\}$) that is preserved by all the diffeomorphisms of G. Then according to the arguments provided in do Carmo to show the existence of a smooth structure on M/G, we know $\{(h_p, U_p, \tilde{V}_p)\}_{p\in M}$ is an atlas for M/G, where $h_p = \pi \circ f_p$, $\tilde{V}_p = \pi(V_p)$, and $\pi: M \longrightarrow M/G$ is the canonical projection. We may assume without loss of generality that $f_a(U_a) \subset W_a$ for some open set $W_a \subset M$ which satisfies $W \cap g(W) = \phi$ for all $g \in G, g \neq e$. Thus $\pi: M \longrightarrow M/G$ is injective when restricted to each $U_p \subset \mathbb{R}^n$. For each pair of $p, q \in M$ such that $\tilde{V}_p \cap \tilde{V}_q \neq \phi$, denote $\pi_p := \pi|_{V_p}: V_p \longrightarrow \tilde{V}_p$, $\pi_q := \pi|_{V_q}: V_q \longrightarrow \tilde{V}_q$. (note that both π_p and π_q are bijective). At each $\tilde{r} \in \tilde{V}_p \cap \tilde{V}_q$, we have

$$h_q^{-1} \circ h_p = f_q^{-1} \circ (\pi_q^{-1} \circ \pi_p) \circ f_p = f_q^{-1} \circ \varphi_g \circ f_p$$

where $g \in G$ satisfies $\varphi_g(p) = q$. Since φ_g preserves the orientation of M, $d(h_q^{-1} \circ h_p) = d(f_q^{-1} \circ \varphi_g \circ f_p)$ has positive Jacobian. Thus M/G is orientable.

"$\Leftarrow$": Assume M/G is orientable. Let $\{(f_a, U_a, \tilde{V}_a)\}$ be an oriented atlas of M/G. It is easy to verify that any refinement of this atlas (with coordinate maps induced by restrictions) is still an oriented atlas. Thus we can assume without loss of generality that $\pi^{-1}(\tilde{V}_a)$ consists of a family of non-overlapping open sets in M. (Otherwise, we can start by constructing an atlas on M such that $\pi(U_a) \subset \tilde{V}_a$ for each index a, and take the projected atlas as the refinement of $\{(f_a, U_a, \tilde{V}_a)\}$, which is orientable.) Denote $\pi^{-1}(\tilde{V}_a) = \bigcup_{\lambda\in\Lambda} V_{a,\lambda}$ where Λ is some index set. Now it is direct to see that $\pi|_{V_{a,\lambda}}$ is a homeomorphism between $V_{a,\lambda}$ and $\tilde{V}_a$ for each $\lambda \in \Lambda$ and each a. Thus we can construct an atlas on M, denoted by $\{(h_{a,\lambda}, U_{a,\lambda}, V_{a,\lambda})\}_{\lambda\in\Lambda}\}$.

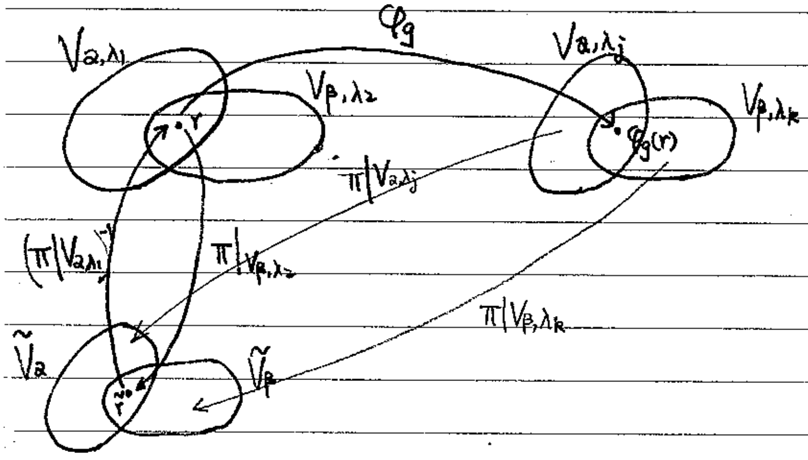
in such a way that $\mathcal{U}_{\alpha,\lambda}=\mathcal{U}_\alpha\subset\mathbb{R}^n$ for all $\lambda\in\Lambda$, $h_{\alpha,\lambda}:=(\pi|V_{\alpha,\lambda})^{-1}\circ f_\alpha:\overset{\mathbb{R}^n}{\overset{\cup}{\mathcal{U}_\alpha}}\to M$.
It is not difficult to verify that $\{(h_{\alpha,\lambda},\mathcal{U}_{\alpha,\lambda},V_{\alpha,\lambda})_{\lambda\in\Lambda}\}$ is an atlas of M, since $V_{\alpha,\lambda_1}\cap V_{\alpha,\lambda_2}=\phi$ for each pair of $\lambda_1\neq\lambda_2$, $\lambda_1,\lambda_2\in\Lambda$, and for each pair of $\alpha\neq\beta$ with arbitrarily chosen $\lambda_1,\lambda_2\in\Lambda$ we know that

$$h_{\beta,\lambda_2}^{-1}\circ h_{\alpha,\lambda_1}=f_\beta^{-1}\circ\left(\pi|_{V_{\beta,\lambda_2}}\right)\circ\left(\pi|_{V_{\alpha,\lambda_1}}\right)^{-1}\circ f_\alpha=f_\beta^{-1}\circ f_\alpha \quad \text{whenever } V_{\alpha,\lambda_1}\cap V_{\beta,\lambda_2}\neq\phi.$$

(Observing that $\left(\pi|V_{\beta,\lambda_2}\right)\circ\left(\pi|V_{\alpha,\lambda_1}\right)^{-1}=Id$ on $V_{\alpha,\lambda_1}\cap V_{\beta,\lambda_2}$) Which gives the required smoothness of $h_{\beta,\lambda_2}^{-1}\circ h_{\alpha,\lambda_1}$. And it follows immediately that

$$\det\left(d(h_{\beta,\lambda_2}^{-1}\circ h_{\alpha,\lambda_1})\right)=\det\left(d(f_\beta^{-1}\circ f_\alpha)\right)>0$$

i.e. $\{(h_{\alpha,\lambda},\mathcal{U}_{\alpha,\lambda},V_{\alpha,\lambda})_{\lambda\in\Lambda}\}$ is an oriented atolas on M.



We are left to show that this orientation is preserved by all the diffeomorphisms of the group G. In fact, for any $g\in G$, the definition of π implies $\pi\circ\varphi_g=\pi$.
Thus $\quad h_{\beta,\lambda_k}^{-1}\circ\varphi_g\circ h_{\alpha,\lambda_1}=f_\beta^{-1}\circ\left(\pi|V_{\beta,\lambda_k}\right)\circ\varphi_g\circ\left(\pi|V_{\alpha,\lambda_1}\right)^{-1}\circ f_\alpha$

$$=f_\beta^{-1}\circ\left[\left(\pi|V_{\beta,\lambda_k}\right)\circ\varphi_g\right]\circ\left(\pi|V_{\alpha,\lambda_j}\right)^{-1}\circ f_\alpha$$

$$=f_\beta^{-1}\circ\left(\pi|V_{\beta,\lambda_2}\right)\circ\left(\pi|V_{\alpha,\lambda_1}\right)^{-1}\circ f_\alpha=f_\beta^{-1}\circ f_\alpha.$$

Hence the orientation constructed is preserved by all the diffeomorphisms of G.

(b) • $\mathbb{RP}^2 = S^2/(\mathbb{Z}/2)$ where the generator of $\mathbb{Z}/2$ acts on S^2 as the antipodal map $x \longmapsto -x$. Obviously this action is properly discontinuous.

Denote $N = (0,0,1) \in \mathbb{R}^3$, $S = (0,0,-1) \in \mathbb{R}^3$. Let $\mathcal{U}_1 = S^2 \setminus \{N\}$, $\mathcal{U}_2 = S^2 \setminus \{S\}$, and the coordinate maps

$$f_1 = \pi_N^{-1} : \mathbb{R}^2 \longrightarrow S^1 : (x_1, x_2) \longmapsto \left(\frac{2x_1}{1+x_1^2+x_2^2}, \frac{2x_2}{1+x_1^2+x_2^2}, \frac{-1+x_1^2+x_2^2}{1+x_1^2+x_2^2} \right)$$

$$f_2 = \pi_S^{-1} : \mathbb{R}^2 \longrightarrow S^1 : (x_1, x_2) \longmapsto \left(\frac{2x_1}{1+x_1^2+x_2^2}, \frac{2x_2}{1+x_1^2+x_2^2}, \frac{1-(x_1^2+x_2^2)}{1+x_1^2+x_2^2} \right)$$

where $\pi_N(y_1,y_2,y_3) = \left(\frac{y_1}{1-y_3}, \frac{y_2}{1-y_3} \right)$, $\pi_S(y_1,y_2,y_3) = \left(\frac{y_1}{1+y_3}, \frac{y_2}{1+y_3} \right)$.

Let 1 denote the generator of $\mathbb{Z}/2$.

φ_1 maps $\mathcal{U}_1$ to $\mathcal{U}_2$ and $\mathcal{U}_2$ to $\mathcal{U}_1$. The induced differential structure is given as $\{(\mathbb{R}^2, \varphi_1 \circ f_1), (\mathbb{R}^2, \varphi_1 \circ f_2)\}$. Note that $\mathcal{U}_2 \cap \mathcal{U}_1 \neq \phi$, thus $f_1^{-1} \circ \varphi_1 \circ f_1 : \mathbb{R}^2 \longrightarrow \mathbb{R}^3$ is well-defined, with

$$\left(f_1^{-1} \circ \varphi_1 \circ f_1\right)(x_1,x_2) = \left(f_1^{-1} \circ \varphi_1\right)\left(\frac{2x_1}{1+x_1^2+x_2^2}, \frac{2x_2}{1+x_1^2+x_2^2}, \frac{-1+x_1^2+x_2^2}{1+x_1^2+x_2^2} \right)$$

$$= f_1^{-1}\left(-\frac{2x_1}{1+x_1^2+x_2^2}, -\frac{2x_2}{1+x_1^2+x_2^2}, \frac{1-(x_1^2+x_2^2)}{1+x_1^2+x_2^2} \right)$$

$$= \left(-\frac{x_1}{x_1^2+x_2^2}, -\frac{x_2}{x_1^2+x_2^2} \right)$$

and thus

$$d\left(f_1^{-1} \circ \varphi_1 \circ f_1\right)(x_1,x_2) = \frac{1}{(x_1^2+x_2^2)^2} \begin{pmatrix} x_1^2 - x_2^2 & 2x_1x_2 \\ 2x_1x_2 & -x_1^2+x_2^2 \end{pmatrix}$$

which gives

$$\det\left(d(f_1^{-1} \circ \varphi_1 \circ f_1)(x_1,x_2) \right) = \frac{1}{(x_1^2+x_2^2)^4}\left[-(x_1^2-x_2^2)^2 - 4x_1^2x_2^2 \right]$$

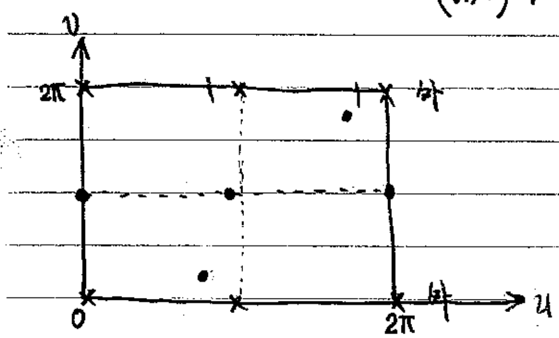
$$= -\frac{1}{(x_1^2+x_2^2)^2} < 0 .$$

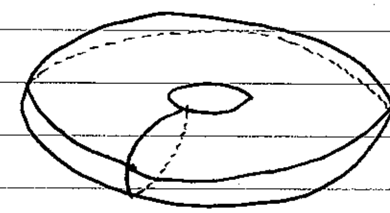
Thus the orientation on S^2 determined by $\{(\mathcal{U}_1, f_1), (\mathcal{U}_2, f_2)\}$ is not preserved by all the diffeomorphisms of $\mathbb{Z}/2$. By a completely identical argument we can see that the other orientation is not preserved by all the diffeomorphisms of $\mathbb{Z}/2$. (This involves only changing a sign of a coordinate.) By (a) we know $\mathbb{RP}^2 = S^2/(\mathbb{Z}/2)$ is not orientable.

Date 2011 12 24

- Klein bottle $= T^2/(\mathbb{Z}/2)$ where the generator of $\mathbb{Z}/2$ acts on T^2 as the antipodal map $x \longmapsto -x$. Obviously this action is properly discontinuous. Take one chart $\mathcal{U}_1 := \{(u,v) \in \mathbb{R}^2 : 0<x<2\pi,\ 0<y<2\pi\}$. The coordinate map is given by

$$f_1 : \mathcal{U}_1 \longrightarrow T^2$$
$$(u,v) \longmapsto \big((R+r\cos v)\cos u,\ (R+r\cos v)\sin u,\ r\sin v\big)$$





By geometric observation we get (note that $(\varphi_1 \circ f_1)(\mathcal{U}_1) \cap \mathcal{U}_1 \neq \phi$)

$$(f_1^{-1} \circ \varphi_1 \circ f_1)(u,v) = f_1^{-1}\big(-(R+r\cos v)\cos u,\ -(R+r\cos v)\sin u,\ r\sin v\big)$$
$$= \begin{cases} (u+\pi,\ 2\pi - v) & \text{if } 0<u<\pi \\ (u-\pi,\ 2\pi-v) & \text{if } \pi<u<2\pi \end{cases}$$

Thus $\det\big(d(f_1^{-1} \circ \varphi_1 \circ f_1)(u,v)\big) = \det\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} = -1 < 0$

which implies that the orientation determined by this atlas is not preserved by all the diffeomorphisms of $\mathbb{Z}/2$. The identical argument works for the other orientations because it only involves a single chart, namely $\mathcal{U}_1$. By (a) we know $K = T^2/(\mathbb{Z}/2)$ is not orientable.

- Möbius band $= \mathbb{R}\times(0,1)/\mathbb{Z}$ where the generator of $\underbrace{\mathbb{Z}}$ (the additive group) acts on $\mathbb{R}\times(0,1)$ as $(x,y) \longmapsto (x+1, 1-y)$. (We can also try the cylinder characterization in Example 4.9(b), but it's not going to be very different from the Klein bottle argument given above.) Take $\mathcal{U}_1 := \mathbb{R}\times(0,1)$ as the only one coordinate charts of $\mathbb{R}\times(0,1)$, and note that the action of the additive group $\mathbb{Z}$ maps $\mathcal{U}_1$ onto itself. (Now we see the subtlety: "smooth manifolds" defined in do Carmo are those without boundaries. So we can only use $\mathbb{R}\times(0,1)$ but not $\mathbb{R}\times[0,1]$.)

No.

Date

Let the coordinate map be the identity: $f: (x,y) \longmapsto (x,y)$. Then

$$\det\left(d\left(f^{-1}\circ\varphi_1\circ f\right)(x,y)\right) = \det\left(d\left(x+1,\, 1-y\right)\right) = \det\begin{pmatrix} 1 & 0 \\ 0 & -1\end{pmatrix} = -1<0.$$

Thus the orientation of $\mathbb{R}\times(0,1)$ is not preserved by all diffeomorphisms of $\mathbb{Z}$. The same situation would occur if we take the other orientation of $\mathcal{U}_1$. Hence by (a) we know $M = \mathbb{R}\times(0,1)/\mathbb{Z}$ is not orientable.

(c) $\mathbb{RP}^n = S^n/(\mathbb{Z}/2)$ where the generator of $\mathbb{Z}/2$ acts on S^n as the antipodal map $x\longmapsto -x$. Obviously this action is properly discontinuous.
Denote $N=(0,\cdots,0,1)\in\mathbb{R}^{n+1}$, $S=(0,\cdots,0,-1)\in\mathbb{R}^{n+1}$. Let $\mathcal{U}_1 = S^n\setminus\{N\}$, $\mathcal{U}_2 = S^n\setminus\{S\}$, and the coordinate maps

$$f_1 = \pi_N^{-1}: \mathbb{R}^n\longrightarrow S^n: (x_1,\cdots,x_n)\longmapsto\left(\frac{2x_1}{1+x_1^2+\cdots+x_n^2},\cdots,\frac{2x_n}{1+x_1^2+\cdots+x_n^2},\frac{-1+x_1^2+\cdots+x_n^2}{1+x_1^2+\cdots+x_n^2}\right)$$

$$f_2 = \pi_S^{-1}: \mathbb{R}^n\longrightarrow S^n: (x_1,\cdots,x_n)\longmapsto\left(\frac{2x_1}{1+x_1^2+\cdots+x_n^2},\cdots,\frac{2x_n}{1+x_1^2+\cdots+x_n^2},\frac{1-(x_1^2+\cdots+x_n^2)}{1+x_1^2+\cdots+x_n^2}\right)$$

Note that $(\varphi_1\circ f_1)(\mathcal{U}_1)\cap\mathcal{U}_1\neq\phi$. Direct computation gives

$$\left(f_1^{-1}\circ\varphi_1\circ f_1\right)(x_1,\cdots,x_n) = \left(f_1^{-1}\circ\varphi_1\right)\left(\frac{2x_1}{1+x_1^2+\cdots+x_n^2},\cdots,\frac{2x_n}{1+x_1^2+\cdots+x_n^2},\frac{-1+x_1^2+\cdots+x_n^2}{1+x_1^2+\cdots+x_n^2}\right)$$

$$= f_1^{-1}\left(-\frac{2x_1}{1+x_1^2+\cdots+x_n^2},\cdots,-\frac{2x_n}{1+x_1^2+\cdots+x_n^2},\frac{1-(x_1^2+\cdots+x_n^2)}{1+x_1^2+\cdots+x_n^2}\right)$$

$$= \left(-\frac{x_1}{x_1^2+\cdots+x_n^2},\cdots,-\frac{x_n}{x_1^2+\cdots+x_n^2}\right)$$

and further

$$d\left(f_1^{-1}\circ\varphi_1\circ f_1\right)(x_1,\cdots,x_n) = \begin{pmatrix} \frac{2x_1^2}{\|x\|^4}-\frac{1}{\|x\|^2} & \frac{2x_1x_2}{\|x\|^4} & \cdots & \frac{2x_1x_n}{\|x\|^4} \\ \frac{2x_1x_2}{\|x\|^4} & \ddots & & \vdots \\ \vdots & & \ddots & \\ \frac{2x_1x_n}{\|x\|^4} & \cdots & & \frac{2x_n^2}{\|x\|^4}-\frac{1}{\|x\|^2}\end{pmatrix},$$

$$= -\frac{1}{\|x\|^4}\left(\|x\|^2 I_n - 2\begin{pmatrix} x_1^2 & x_1x_2 & \cdots & x_1x_n \\ x_1x_2 & \ddots & & \vdots \\ \vdots & & \ddots & \\ x_1x_n & \cdots & & x_n^2\end{pmatrix}\right)$$

where $\|x\|^2 = x_1^2+\cdots+x_n^2$
and I_n stands for the $n\times n$ identity matrix.

Thus

$$\det\left(d(f_1^{-1}\circ\varphi_1\circ f_1)(x_1,\cdots,x_n)\right)=\frac{(-1)^n}{\|x\|^{2n}}\cdot\det\left(I_n-\frac{\sqrt2}{\|x\|}\begin{pmatrix}x_1\\ \vdots\\ x_n\end{pmatrix}\cdot\frac{\sqrt2}{\|x\|}(x_1,\cdots,x_n)\right)$$

$$=\frac{(-1)^n}{\|x\|^{2n}}\cdot\det\left(1-\left(\frac{\sqrt2}{\|x\|}\right)^2(x_1,\cdots,x_n)\begin{pmatrix}x_1\\ \vdots\\ x_n\end{pmatrix}\right)$$

$$=\frac{(-1)^n}{\|x\|^{2n}}\cdot\det\left(1-\frac{2}{\|x\|^2}\cdot\|x\|^2\right)=\frac{(-1)^{n+1}}{\|x\|^{2n}}.$$

- If n is even, then $\det\left(d(f_1^{-1}\circ\varphi_1\circ f_1)(x_1,\cdots,x_n)\right)=\frac{(-1)^{n+1}}{\|x\|^{2n}}<0$, and the orientation on S^n determined by $\{(\mathcal{U}_1,f_1),(\mathcal{U}_2,f_2)\}$ is not preserved by all diffeomorphisms of $\mathbb{Z}/2$. By (a) we know $\mathbb{RP}^n$ (with n even) is not orientable.
- If n is odd, then $\det\left(d(f_1^{-1}\circ\varphi_1\circ f_1)(x_1,\cdots,x_n)\right)=\frac{(-1)^{n+1}}{\|x\|^{2n}}>0$. Note that there are only two coordinate charts ($\mathcal{U}_1$ and $\mathcal{U}_2$) on S^n, $\varphi_1(\mathcal{U}_1)=\mathcal{U}_2$ and $\varphi_1(\mathcal{U}_2)=\mathcal{U}_1$, thus $d(\tilde f_2^{-1}\circ\varphi_1\circ f_1)(x_1,\cdots,x_n)=d\left((\tilde f_2^{-1}\circ f_1\circ f_1^{-1}\circ\varphi_1\circ f_1)\right)(x_1,\cdots,x_n)$

$$=\left(d(\tilde f_2^{-1}\circ f_1)\cdot d(f_1^{-1}\circ\varphi_1\circ f_1)\right)(x_1,\cdots,x_n)$$

also has positive determinant because both $d(\tilde f_2^{-1}\circ f_1)$ and $d(f_1^{-1}\circ\varphi_1\circ f_1)$ do. Here $\tilde f_2$ differs from f_2 by only one sign in exactly one coordinate. By a similar argument we can check that both $d(\tilde f_2^{-1}\circ\varphi_1\circ\tilde f_2)$ and $d(f_1^{-1}\circ\varphi_1\circ\tilde f_2)$ have positive determinant (or by observation that $d(\tilde f_2^{-1}\circ\varphi_1\circ\tilde f_2)=d(\tilde f_2^{-1}\circ f_1)\,d(f_1^{-1}\circ\varphi_1\circ f_1)\,d(f_1^{-1}\circ\tilde f_2)$ on $\tilde f_2(\mathbb{R}^n)\cap(\varphi_1\circ\tilde f_2)(\mathbb{R}^n)=\mathcal{U}_2\cap\mathcal{U}_1=(\varphi_1\circ f_1)(\mathbb{R}^n)\cap f_1(\mathbb{R}^n)$, and that $d(f_1^{-1}\circ\varphi_1\circ\tilde f_2)=$ $=\left(d(\tilde f_2^{-1}\circ\varphi_1\circ f_1)\right)^{-1}$ on $\mathcal{U}_1\cap\mathcal{U}_2$ together with the observation that $d(f_1^{-1}\circ\varphi_1\circ\tilde f_2)$ does not change sign on the whole $\mathcal{U}_1\supset\mathcal{U}_1\cap\mathcal{U}_2$ and that $\varphi_1^{-1}=\varphi_1$; if this long remark makes anything easier.). Hence the orientation of S^n is preserved by all the diffeomorphisms of $\mathbb{Z}/2$, and it follows from (a) that $\mathbb{RP}^n$ is orientable when n is odd.
- Combining both item's above, we conclude that $\mathbb{RP}^n$ is orientable if and only if n is odd.

Remark: There is no redundant arguments in (b): we do have to check for "all" possible orientations on the covering space.

No.

Date

10. Show that the topology of the differentiable manifold M/G of Example 4.8 is Hausdorff if and only if the following condition holds: given two non-equivalent points $p_1, p_2 \in M$, there exist neighborhoods $\mathcal{U}_1, \mathcal{U}_2$ of p_1 and p_2, respectively, such that $\mathcal{U}_1 \cap g\mathcal{U}_2 = \phi$ for all $g \in G$.

- Proof: "$\Rightarrow$": Assume M/G is Hausdorff. Given two non-equivalent points $p_1, p_2 \in M$, we have $\pi(p_1) \neq \pi(p_2)$. Thus there exists neighborhoods $\tilde{\mathcal{U}}_1, \tilde{\mathcal{U}}_2$ in M/G of $\pi(p_1)$ and $\pi(p_2)$ respectively, such that $\tilde{\mathcal{U}}_1 \cap \tilde{\mathcal{U}}_2 = \phi$. Then $\pi^{-1}(\tilde{\mathcal{U}}_1) =: \mathcal{U}_1$ and $\pi^{-1}(\tilde{\mathcal{U}}_2) =: \mathcal{U}_2$ satisfies $\mathcal{U}_1 \cap g\mathcal{U}_2 = \phi$ for all $g \in G$, for otherwise we would have $\tilde{\mathcal{U}}_1 \cap \tilde{\mathcal{U}}_2 \neq \phi$. (If we agree to shrink $\tilde{\mathcal{U}}_1$ and $\tilde{\mathcal{U}}_2$ sufficiently small, we can find connected open neighborhoods $\bar{\mathcal{U}}_1 \ni p_1$, $\bar{\mathcal{U}}_2 \ni p_2$ such that $\pi|_{\bar{\mathcal{U}}_1}: \bar{\mathcal{U}}_1 \to \tilde{\mathcal{U}}_1$ and $\pi|_{\bar{\mathcal{U}}_2}: \bar{\mathcal{U}}_2 \to \tilde{\mathcal{U}}_2$ are both diffeomorphisms. Then $\pi^{-1}(\tilde{\mathcal{U}}_1) = \bigcup_{g \in G} g\bar{\mathcal{U}}_1$, $\pi^{-1}(\tilde{\mathcal{U}}_2) = \bigcup_{g \in G} g\bar{\mathcal{U}}_2$, thus $g(\pi^{-1}(\tilde{\mathcal{U}}_2)) = \pi^{-1}(\tilde{\mathcal{U}}_2)$ and hence $\pi^{-1}(\tilde{\mathcal{U}}_1) \cap g(\pi^{-1}(\tilde{\mathcal{U}}_2)) = \phi$.

"$\Leftarrow$": For $\tilde{p}_1, \tilde{p}_2 \in M/G$ such that $\tilde{p}_1 \neq \tilde{p}_2$, pick $p_1 \in \pi^{-1}(\tilde{p}_1)$ and $p_2 \in \pi^{-1}(\tilde{p}_2)$. Then p_1 and p_2 are non-equivalent. By assumption, we can take neighborhoods $\mathcal{U}_1, \mathcal{U}_2$ of p_1 and p_2 respectively such that $\mathcal{U}_1 \cap g\mathcal{U}_2 = \phi$ for all $g \in G$. We claim that $\pi(\mathcal{U}_1) \cap \pi(\mathcal{U}_2) = \phi$. In fact, assume $\tilde{r} \in \pi(\mathcal{U}_1) \cap \pi(\mathcal{U}_2)$, then $\pi(r_1) = \tilde{r}$ and $\pi(r_2) = \tilde{r}$ for some $r_1, r_2 \in M$, and $r_1 = g_1 r$ for some $r \in \mathcal{U}_1, g_1 \in G$. Then $g_1 r = r_1 = g_2 r_2$ for some $g_2 \in G$ because $\pi(r_1) = \pi(r_2)$. It follows that $\mathcal{U}_1 \ni r = g_1^{-1} g_2 r_2 \in (g_1^{-1} g_2)\mathcal{U}_2$, contradicting $\mathcal{U}_1 \cap g\mathcal{U}_2 = \phi$ for all $g \in G$. Finally, note that both $\pi(\mathcal{U}_1)$ and $\pi(\mathcal{U}_2)$ are open sets in G/M, and $\tilde{p}_1 \in \pi(\mathcal{U}_1)$, $\tilde{p}_2 \in \pi(\mathcal{U}_2)$, we can conclude that the topology of M/G is Hausdorff.

11. Let us consider the two following differentiable structures on the real line $\mathbb{R}$: $(\mathbb{R}, x_1)$, where $x_1 : \mathbb{R} \to \mathbb{R}$ is given by $x_1(x) = x$, $x \in \mathbb{R}$; $(\mathbb{R}, x_2)$, where $x_2 : \mathbb{R} \to \mathbb{R}$ is given by $x_2(x) = x^3$, $x \in \mathbb{R}$. Show that:
(a) the identity mapping $i : (\mathbb{R}, x_1) \to (\mathbb{R}, x_2)$ is not a diffeomorphism; therefore, the maximal structures determined by $(\mathbb{R}, x_1)$ and $(\mathbb{R}, x_2)$ are distinct.
(b) the mapping $f : (\mathbb{R}, x_1) \to (\mathbb{R}, x_2)$ given by $f(x) = x^3$ is a diffeomorphism; that is, even though the differentiable structure $(\mathbb{R}, x_1)$ and $(\mathbb{R}, x_2)$ are distinct, they determine diffeomorphic differentiable manifolds.

- Proof: (a) $x_2^{-1} \circ i \circ x_1 : \mathbb{R} \to \mathbb{R}$, $x \mapsto x^{1/3}$ is not differentiable at $x = 0$.

(b) $x_2^{-1} \circ f \circ x_1 : \mathbb{R} \to \mathbb{R}$, $x \mapsto x$ is differentiable on $\mathbb{R}$.

$f : \mathbb{R} \to \mathbb{R}$ is bijective, with $f^{-1} : \mathbb{R} \to \mathbb{R} : x \mapsto x^{1/3}$.
$x_1^{-1} \circ f^{-1} \circ x_2 : \mathbb{R} \to \mathbb{R}$, $x \mapsto x$ is also differentiable on $\mathbb{R}$.

Thus the mapping $f : (\mathbb{R}, x_1) \to (\mathbb{R}, x_2)$ given by $f(x) = x^3$ is a diffeomorphism.

No.

Date

12. (The orientable double covering). Let M^n be a non-orientable differentiable manifold. For each $p \in M$, consider the set B of bases of T_pM and say that the two bases are equivalent if they are related by a matrix with positive determinant. This is an equivalence relation and separates B into two disjoint sets. Let Θ_p be the quotient space of B with respect to this equivalence relation. $O_p \in \Theta_p$ will be called an orientation of T_pM. Let $\overline{M}$ be the set $\overline{M} = \{(p, O_p);\, p \in M,\, O_p \in \Theta_p\}$. Let $\{(\mathcal{U}_\alpha, x_\alpha)\}$ be a $\boxed{\text{maximal}}$ differentiable structure on M, and define $\overline{x}_\alpha : \mathcal{U}_\alpha \to \overline{M}$ by

$$\overline{x}_\alpha(u_1^\alpha, \cdots, u_n^\alpha) = \left(x_\alpha(u_1^\alpha, \cdots, u_n^\alpha),\ \left[\frac{\partial}{\partial u_1^\alpha}, \cdots, \frac{\partial}{\partial u_n^\alpha}\right] \right),$$

where $(u_1^\alpha, \cdots, u_n^\alpha) \in \mathcal{U}_\alpha$ and $\left[\frac{\partial}{\partial u_1^\alpha}, \cdots, \frac{\partial}{\partial u_n^\alpha}\right]$ denotes the element of Θ_p determined by the basis $\left\{\frac{\partial}{\partial u_1^\alpha}, \cdots, \frac{\partial}{\partial u_n^\alpha}\right\}$. Prove that:

(a) $\{(\mathcal{U}_\alpha, \overline{x}_\alpha)\}$ is a differentiable structure on $\overline{M}$ and that the manifold $\overline{M}$ so obtained is orientable.

(b) The mapping $\pi : \overline{M} \to M$ given by $\pi(p, O_p) = p$ is differentiable and surjective. In addition, each $p \in M$ has a neighborhood $\mathcal{U} \subset M$ such that $\pi^{-1}(\mathcal{U}) = V_1 \cap V_2$, where V_1 and V_2 are disjoint open sets in $\overline{M}$ and π restricted to each V_i, $i = 1, 2$, is a diffeomorphism onto $\mathcal{U}$. For this reason, $\overline{M}$ is called the orientable double cover of M.

(c) The sphere S^2 is the orientable double cover of $\mathbb{RP}^2$ and the torus T^2 is the orientable double cover of the Klein bottle K.

- Proof: (a) $\boxed{\text{Obviously}}$ $\bigcup_\alpha \overline{x}_\alpha(\mathcal{U}_\alpha) \supset \overline{M}$. If $\overline{x}_\alpha(\mathcal{U}_\alpha) \cap \overline{x}_\beta(\mathcal{U}_\beta) \neq \phi$, then at any $(p, O_p) \in \overline{x}_\alpha(\mathcal{U}_\alpha) \cap \overline{x}_\beta(\mathcal{U}_\beta)$ we have $x_\alpha(u_1^\alpha, \cdots, u_n^\alpha) = p = x_\beta(u_1^\beta, \cdots, u_n^\beta)$ and $[\partial/\partial u_1^\alpha, \cdots, \partial/\partial u_n^\alpha] = [\partial/\partial u_1^\beta, \cdots, \partial/\partial u_n^\beta]$. Thus $\det\left(\frac{\partial u_j^\beta}{\partial u_k^\alpha}\right) > 0$. Now we have $(\overline{x}_\beta^{-1} \circ \overline{x}_\alpha)(u_1^\alpha, \cdots, u_n^\alpha) = (u_1^\beta, \cdots, u_n^\beta)$ smooth because $\overline{x}_\alpha(\mathcal{U}_\alpha) \cap \overline{x}_\beta(\mathcal{U}_\beta) \neq \phi$ implies that $x_\alpha(\mathcal{U}_\alpha) \cap x_\beta(\mathcal{U}_\beta) = \phi$ and $(x_\beta^{-1} \circ x_\alpha)(u_1^\alpha, \cdots, u_n^\alpha) = (u_1^\beta, \cdots, u_n^\beta)$ is smooth. Hence $\{(\mathcal{U}_\alpha, \overline{x}_\alpha)\}$ is a differentiable structure on $\overline{M}$. Moreover, $\det(d(\overline{x}_\beta^{-1} \circ \overline{x}_\alpha)) = \det\left(\frac{\partial u_k^\beta}{\partial u_j^\alpha}\right) > 0$ since $\det\left(\frac{\partial u_j^\alpha}{\partial u_k^\beta}\right) > 0$. Thus $\{(\mathcal{U}_\alpha, \overline{x}_\alpha)\}$ is an oriented atlas of $\overline{M}$, and it follows by definition that $\overline{M}$ is orientable.

Not really obvious. Here we need the maximality of this differential structure. For any $p \in M$, take $(\mathcal{U}_\alpha, x_\alpha)$ and $(\mathcal{U}_\beta, x_\beta)$ such that $\det(d(x_\beta^{-1} \circ x_\alpha)) < 0$ and $p \in x_\alpha(\mathcal{U}_\alpha) \cap x_\beta(\mathcal{U}_\beta)$. Then $[\partial/\partial u_1^\alpha, \cdots, \partial/\partial u_n^\alpha] = -[\partial/\partial u_1^\beta, \cdots, \partial/\partial u_n^\beta]$ while $x_\alpha(u_1^\alpha, \cdots, u_n^\alpha) = p = x_\beta(u_1^\beta, \cdots, u_n^\beta)$. Thus $\bigcup_\alpha \overline{x}_\alpha(\mathcal{U}_\alpha) \supset \overline{M}$.

(b) That $\pi : \bar{M} \to M$ is onto is obvious. To see that π is differentiable, take coordinate chart (U_α, x_α) near $p \in M$ and coordinate chart $(U_\alpha, \bar{x}_\alpha)$ near $(p, O_p) \in \bar{M}$. Then $x_\alpha^{-1} \circ \pi \circ \bar{x}_\alpha : U_\alpha \to U_\alpha$ is given by $(u_1^\alpha, \cdots, u_n^\alpha) \mapsto (u_1^\alpha, \cdots, u_n^\alpha)$, which is obviously differentiable.

Now let $p \in M$ be an arbitrary point in M. Take $U \subset M$ to be an open neighborhood of p. By the maximality of the differentiable structure $\{(U_\alpha, x_\alpha)\}$, we can take coordinate charts (U_α, x_α) and (U_β, x_β) around p such that the differential of $x_\beta^{-1} \circ x_\alpha : U_\alpha \to U_\beta$ has negative determinant at each point $q \in U_\alpha$. By possibly shrinking U_α, U_β and U, we may assume without loss of generality that $x_\alpha(U_\alpha) = U = x_\beta(U_\beta)$. Now define

$$V_1 = \bar{x}_\alpha(x_\alpha^{-1}(U)) = \bar{x}_\alpha(U_\alpha), \quad V_2 = \bar{x}_\beta(x_\beta^{-1}(U)) = \bar{x}_\beta(U_\beta),$$

then $V_1 \subset \bar{M}$, $V_2 \subset \bar{M}$, and V_1 and V_2 are both open sets in $\bar{M}$ because the topology on $\bar{M}$ is induced by the differential structure $\{(U_\alpha, \bar{x}_\alpha)\}$ (see Remark 2.3 on pp.3). It is also direct to see that $V_1 \cap V_2 = \phi$, for otherwise if $q \in V_1 \cap V_2$ then $x_\alpha(v_1^\alpha, \cdots, v_n^\alpha) = q = x_\beta(v_1^\beta, \cdots, v_n^\beta)$ and $[\partial/\partial v_1^\alpha, \cdots, \partial/\partial v_n^\alpha] = [\partial/\partial v_1^\beta, \cdots, \partial/\partial v_n^\beta]$, which contradicts the assumption that

$$\det\left(\frac{\partial v_j^\beta}{\partial v_k^\alpha}\right) = \det\left(d(x_\beta^{-1} \circ x_\alpha)\Big|_{x_\alpha^{-1}(q)}\right) < 0.$$

Since at each $p \in M$ there exist two and only two orientations for T_pM, we know actually $\pi^{-1}(U) = V_1 \cup V_2$. (for each $p \in U$, $\pi^{-1}(p) = \{(p, O_p), (p, -O_p)\}$.) Also, it is direct to see that the definition of V_1 and V_2 is independent of the choice of coordinates: if (U_γ, x_γ) is another coordinate chart around p with $\det(d(x_\gamma^{-1} \circ x_\alpha)) > 0$ on $x_\alpha^{-1}(x_\alpha(U_\alpha) \cap x_\gamma(U_\gamma))$, then $\bar{x}_\gamma(x_\gamma^{-1}(U \cap x_\gamma(U_\gamma))) \subseteq V_1$. (A similar statement holds for (U_β, x_β).) Even better, whenever $x_\gamma(u_1^\gamma, \cdots, u_n^\gamma) = x_\alpha(u_1^\alpha, \cdots, u_n^\alpha)$, we have $\bar{x}_\gamma(u_1^\gamma, \cdots, u_n^\gamma) = \bar{x}_\alpha(u_1^\alpha, \cdots, u_n^\alpha)$. Thus we are free to use a fixed coordinate system around p. (Similar for (U_β, x_β).)

Moreover, $\pi|V_1 : V_1 \to U$ is bijective. First it is obviously onto. Then for any point $q \in U$, there is a unique point $x_\alpha^{-1}(q) \in U_\alpha$, and then a unique point $\bar{x}_\alpha(x_\alpha^{-1}(q)) \in V_1$ correspondingly. Hence $\pi|V_1$ is invertible. It has been shown above that $\pi|V_1$ is differentiable. For $(\pi|V_1)^{-1}$, note that $(\bar{x}_\alpha^{-1} \circ (\pi|V_1)^{-1} \circ x_\alpha)(u_1^\alpha, \cdots, u_n^\alpha) = (u_1^\alpha, \cdots, u_n^\alpha)$, which is the identity map and thus is differentiable. Therefore $\pi|V_1 : V_1 \to U$ is a diffeomorphism. Similarly we know that $\pi|V_2 : V_2 \to U$ is a diffeomorphism.

No.

Date

(c) • First we want to show the following: If M is connected, then $\overline{M}$ is connected if and only if M is non-orientable. Equivalently, it suffices to prove that if M is connected, then $\overline{M}$ is disconnected if and only if M is orientable.

"$\Leftarrow$": If M is orientable, then by the connectedness there exist two and only two distinct orientations. Let $\{(V_\alpha^+, x_\alpha^+)\}$ and $\{(V_\beta^-, x_\beta^-)\}$ be two oriented atlases determining two distinct orientations. Let

$$\tilde{V}^+ := \bigcup_\alpha \overline{x_\alpha^+}(V_\alpha^+) \quad , \quad \tilde{V}^- := \bigcup_\beta \overline{x_\beta^-}(V_\beta^-).$$

Then $\tilde{V}^+ \cup \tilde{V}^- \supset \overline{M}$, $\tilde{V}^+$ and $\tilde{V}^-$ are both open sets in $\overline{M}$, and $\tilde{V}^+ \cap \tilde{V}^- = \phi$.

In fact, for any $(p, O_p) \in \overline{M}$, $p \in M$ and there exists (V_p^+, x_p^+) and (V_p^-, x_p^-) around $p \in M$, belonging to $\{(V_\alpha^+, x_\alpha^+)\}$ and $\{(V_\beta^-, x_\beta^-)\}$ respectively. Then

$\overline{x_p^+}(u_1^+(p), \cdots, u_n^+(p)) = \left(x_p^+(u_1^+(p), \cdots, u_n^+(p)), \left[\frac{\partial}{\partial u_1^+}\Big|_p, \cdots, \frac{\partial}{\partial u_n^+}\Big|_p\right]\right)$ and

$\overline{x_p^-}(u_1^-(p), \cdots, u_n^-(p)) = \left(x_p^-(u_1^-(p), \cdots, u_n^-(p)), \left[\frac{\partial}{\partial u_1^-}\Big|_p, \cdots, \frac{\partial}{\partial u_n^-}\Big|_p\right]\right)$ determine different points in $\overline{M}$ lying on the same fiber. Thus either

$\overline{x_p^+}(u_1^+(p), \cdots, u_n^+(p)) = (p, O_p)$ or $\overline{x_p^-}(u_1^-(p), \cdots, u_n^-(p)) = (p, -O_p)$. This proves $\tilde{V}^+ \cup \tilde{V}^- \supset \overline{M}$.

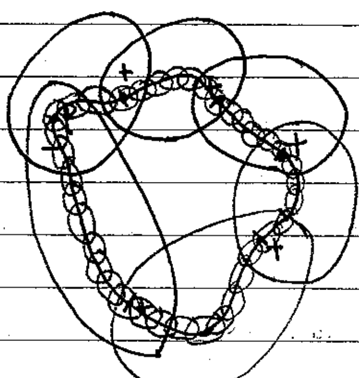
That $\tilde{V}^+$ and $\tilde{V}^-$ are both open sets in $\overline{M}$ follows from the construction of $\overline{M}$.

If $\tilde{V}^+ \cap \tilde{V}^- \neq \phi$, say $(q, O_q) \in \tilde{V}^+ \cap \tilde{V}^-$, then there exist neighborhoods U_q^+, U_q^- around q in M such that $(q, O_q) \in \overline{x_q^+}(U_q^+) \cap \overline{x_q^-}(U_q^-)$. Thus O_q is compatible with both $\left[\frac{\partial}{\partial u_1^+}\Big|_q, \cdots, \frac{\partial}{\partial u_n^+}\Big|_q\right]$ and $\left[\frac{\partial}{\partial u_1^-}\Big|_q, \cdots, \frac{\partial}{\partial u_n^-}\Big|_q\right]$, which is impossible. Thus $\tilde{V}^+ \cap \tilde{V}^- = \phi$.

Moreover, since for any $p \in M$, (p, O_p) and $(p, -O_p)$ can't both sit on either $\tilde{V}^+$ or $\tilde{V}^-$, thus $\overline{M} \cap \tilde{V}^+ \neq \phi$ and $\overline{M} \cap \tilde{V}^- \neq \phi$. This proves that $\overline{M}$ is disconnected.

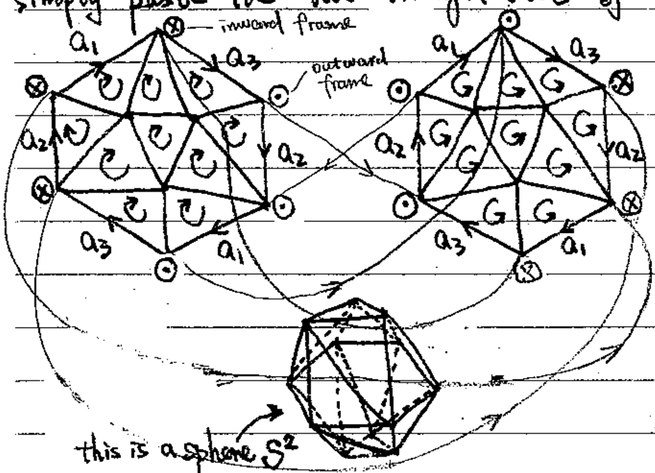
"$\Rightarrow$": If M is non-orientable, we know there exists a "contradictory chain" of charts on M (cf. Zorich, Vol II, §15.2.3 Proposition 4), i.e. there exists a finite sequence of charts of the atlas $\{(U_\alpha, x_\alpha)\}$, denoted by $\{(U_1, x_1), \cdots, (U_m, x_m)\}$ such that $U_j \cap U_{j+1} \neq \phi$, $1 \le j \le m-1$, $\det(d(x_{j+1}^{-1} \circ x_j)) > 0$, $1 \le j \le m+1$, and $U_m \cap U_1 \neq \phi$ with $\det(d(x_1^{-1} \circ x_m)) < 0$. By the connectedness of $\bigcup_{k=1}^m U_k$ and the local connectedness of each U_k $(1 \le k \le m)$, we know $\bigcup_{k=1}^m U_k$ is path connected.

In particular, for some $p \in \mathcal{U}_m \cap \mathcal{U}_1$ we can find a path starting from p while ending in p, with the entire path lying in $\bigcup_{k=1}^{m} \mathcal{U}_k$. Split each $\mathcal{U}_k$ into smaller pieces of open sets if necessary such that the inverse image of each piece under the projection π consists of two disjoint open sets in $\bar{M}$ (as described in (b)), and use the compactness of each closed segment of the path to extract a finite collection of such pieces. All the redundancies we put here is to lift the path to a connected path in $\bar{M}$, which connects (p, O_p) and $(p, -O_p)$. Here $\bar{x}_1(u_1^1(p), \cdots, u_n^1(p))$ is different from $\bar{x}_m(u_1^m(p), \cdots, u_n^m(p))$ because $\det(d(x_m^{-1} \circ x_1)) < 0$. The compactness of the loop in M comes from the compactness of the interval $[0,1]$. Now for any $q \in M$, $q \neq M$, we can construct a chain of coordinate charts $\{V_1, \cdots, V_{m(q)}\}$ such that $p \in V_1$, $q \in V_{m(q)}$, $V_j \cap V_{j+1} \neq \phi$ for $1 \le j \le m(q)-1$, and $\det(d(x_{V_j}^{-1} \circ x_{V_{j+1}})) > 0$ for $1 \le j \le m(q)-1$. This can be done as follows: first find a path lying in M connecting p to q, then use the compactness to extract a finite open cover, and then adjust the signs accordingly. To avoid contradiction we may want to shrink V_1 and $V_{m(q)}$ appropriately such that $V_1 \cap V_{m(q)} = \phi$ while keeping $p \in V_1$, $q \in V_{m(q)}$. (This can be done because the topology on M is Hausdorff.) Then by a similar argument as before (split each V_k and take finite splitted pieces to cover the path) we can lift this path to $\bar{M}$. This connects (q, O_q) to either (p, O_p) or $(p, -O_p)$, or connects $(q, -O_q)$ to either (p, O_p) or $(p, -O_p)$. By switching the sign of exactly one coordinate we can connect both (q, O_q) and $(q, -O_q)$ to either (p, O_p) or $(p, -O_p)$. Since (p, O_p) and $(p, -O_p)$ are pathly connected, we have shown that any point $(q, \pm O_q)$ is pathly connected to (p, O_p). Hence $\bar{M}$ is path connected, and hence is connected.



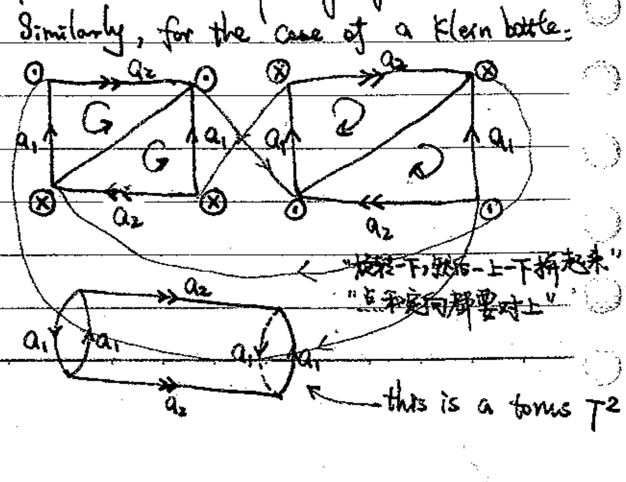
- Next we show that the sphere S^2 is the orientable double cover of $\mathbb{RP}^2$. By the "Galois Correspondence" between conjugacy classes of subgroups of $\pi_1(\mathbb{RP}^2)$ and isomorphism classes of path-connected covering spaces (no need to fix a base point because $\mathbb{RP}^2$ is connected) (see Hatcher, P67-68 Theorem 1.38 for this correspondence), and the fact that $\pi_1(\mathbb{RP}^2) \cong \mathbb{Z}/2\mathbb{Z}$, we know $\mathbb{RP}^2$ has a unique pathwise connected double cover modulo homeomorphisms. Since as a manifold, $\mathbb{RP}^2$ is locally path-connected, so is the unique pathwise connected double cover. A locally path-connected and path-connected topological space is connected. Hence $\mathbb{RP}^2$ has a unique connected double cover. Note that S^2 is a connected double cover of $\mathbb{RP}^2$, thus it is the unique one. Miraculously it is orientable, thus S^2 is the orientable double cover of $\mathbb{RP}^2$.

- Then we show that the torus T^2 is the orientable double cover of the Klein bottle K. Note that the orientable double cover of K is connected, and its Euler characteristic is $2\chi(K) = 2\times 0 = 0$. Thus the orientable double cover of K is homeomorphically equivalent to a connected orientable two-dimensional manifold. (That it is a manifold is obvious. Just make use of the coordinate charts on K and locally pull them back to the orientable double cover. Moreover, since K is compact, its orientable double cover is also compact. Since K has no boundary, neither does its orientable double cover. Now by the classification theorem of closed (compact, connected, without boundary) surfaces (two-dimensional real manifolds) we know the orientable double cover of the Klein bottle K is a closed orientable with genus 1. In other words, it is homeomorphic to the torus T^2. (This argument which based on the Euler characteristic can also be applied to the $\mathbb{RP}^2$ case)
- Finally we make the remark here that all we have done so far for problem 12(c) maybe completely redundant. To find the orientable double cover of $\mathbb{RP}^2$, simply paste the two triangulations of $\mathbb{RP}^2$ with distinct orientations:



(use frame to define orientations on each point, thus can temporarily ignore the arrows.)

Similarly, for the case of a Klein bottle.



1. Prove that the antipodal mapping $A: S^n \longrightarrow S^n$ given by $A(p) = -p$ is an isometry of S^n. Use this fact to introduce a Riemannian metric on the real projective space $\mathbb{RP}^n$ such that the natural projection $\pi: S^n \longrightarrow \mathbb{RP}^n$ is a local isometry.

- Proof: 1°. • We show the following very useful observation: for all $p \in S^n$, there is a characterization of $T_pS^n := \{x \in \mathbb{R}^{n+1} : \langle x, p\rangle = 0\}$. Here $\langle , \rangle$ denotes for the standard Euclidean metric on $\mathbb{R}^{n+1}$. To see this, let $S^n := \{(x_0, \cdots, x_n) \in \mathbb{R}^{n+1} : x_0^2 + \cdots + x_n^2 = 1\}$. For an arbitrary curve $\gamma: (-\varepsilon, \varepsilon) \longrightarrow S^n$, let $\gamma(t) := (\gamma_0(t), \cdots, \gamma_n(t))$ for $t \in (-\varepsilon, \varepsilon)$, then $(\gamma_0(t))^2 + \cdots + (\gamma_n(t))^2 = 1$. Assume $\gamma(0) = p$. Then for any $f \in C^\infty(S^n)$,
$$\gamma'(0)f = \frac{d}{dt}\Big|_{t=0} f(\gamma(t)) = \frac{d}{dt}\Big|_{t=0} f(\gamma_0(t), \cdots, \gamma_n(t)) = \sum_{k=0}^n \frac{\partial f}{\partial x_k}\Big|_p \gamma_k'(0) = \left(\sum_{k=0}^n \gamma_k'(0)\frac{\partial}{\partial x_k}\Big|_p\right) f.$$
So $\gamma'(0) = (\gamma_0'(0), \cdots, \gamma_n'(0)) \in \mathbb{R}^{n+1}$, as we already knew. By $(\gamma_0(t))^2 + \cdots + (\gamma_n(t))^2 = 1$ we know that $2\sum_{k=0}^n \gamma_k(0)\gamma_k'(0) = 0 \iff \langle \gamma'(0), p\rangle = \langle \gamma'(0), \gamma(0)\rangle = 0$. Thus $\langle \gamma'(0), p\rangle = 0$. On the other hand, for any $v \in \mathbb{R}^{n+1}$ satisfying $\langle v, p\rangle = 0$, we can find a curve $\gamma: (-\varepsilon,\varepsilon) \to \mathbb{R}^{n+1}$ such that $\gamma(0) = p$ and $\gamma(t) \subset S^n$ for all $t \in (-\varepsilon, \varepsilon)$, with $\gamma'(0) = v$. This is because we can take γ to be a big circle on S^n lying in the plane containing v and the origin. For example, we can take $\gamma(t) = \cos(\|v\|t)p + \sin(\|v\|t)\frac{v}{\|v\|}$. Then $\langle p, v\rangle = 0$ gives $\langle \gamma(t), \gamma(t)\rangle \equiv 1$ $\forall t \in (-\varepsilon, \varepsilon)$, and it is easy to verify that $\gamma(0) = p$, $\gamma'(0) = v$. Hence we have shown that $T_pS^n = \{x \in \mathbb{R}^{n+1} : \langle x, p\rangle = 0\}$.

- Keeping with the notations above, we have that
$$\left(dA_p(\gamma'(0))\right)f = \frac{d}{dt}\Big|_{t=0}(f \circ A \circ \gamma)(t) = \frac{d}{dt}\Big|_{t=0} f(-\gamma(t)) = \frac{d}{dt}\Big|_{t=0} f(-\gamma_0(t), \cdots, -\gamma_n(t))$$
$$= \sum_{k=0}^n -\frac{\partial f}{\partial x_k}\Big|_{-p}\gamma_k'(0) = \left(\sum_{k=0}^n (-\gamma_k'(0))\frac{\partial}{\partial x_k}\Big|_{-p}\right) f$$
which implies that $dA_p: T_pS^n \longrightarrow T_{-p}S^n$ is given by $\gamma'(0) \longmapsto -\gamma'(0)$.

- A faster way to see what dA_p is can be given as follows (using the observation at the beginning of this proof): Since $A: S^n \to S^n$ can be viewed as the restriction of $\tilde{A}: \mathbb{R}^{n+1} \longrightarrow \mathbb{R}^{n+1}$ to S^n, we want to say that $dA_p: T_pS^n \longrightarrow T_{-p}S^n$ is just the restriction of $d\tilde{A}_p: T_p\mathbb{R}^{n+1} \longrightarrow T_p\mathbb{R}^{n+1}$. Note that $A = \tilde{A} \circ i: S^n \longrightarrow \mathbb{R}^n$ where i is the inclusion map from S^n to $\mathbb{R}^{n+1}$. Thus $dA_p = d\tilde{A}_p \cdot di_p: T_pS^n \to T_{-p}\mathbb{R}^n$. But di_p is just the identity, and by our previous observation we can see that $T_pS^n = \{x \in \mathbb{R}^{n+1} : \langle x, p\rangle = 0\}$ is mapped by $d\tilde{A}_p$ into $T_{-p}S^n = \{x \in \mathbb{R}^{n+1} : \langle x, -p\rangle = 0\}$ (because $d\tilde{A}_p = \tilde{A}$ if we identify $T_p\mathbb{R}^{n+1}$ with $\mathbb{R}^{n+1}$, since $\tilde{A}$ is linear). Thus $dA_p = d\tilde{A}_p\big|_{T_pS^n}$.

No.

Date

Or more precisely, $A = r \circ \tilde{A} \circ i : S^n \longrightarrow S^n$, where $i : S^n \to \mathbb{R}^{n+1}$ is the inclusion map, $\tilde{A} : \mathbb{R}^{n+1} \longrightarrow \mathbb{R}^{n+1}$ is the antipodal map, and $r : \mathbb{R}^{n+1} \longrightarrow S^n$ is the renormalization map $x \mapsto x/\|x\|$ for $x \neq 0$ and $x \mapsto 0$ for $x = 0$. Thus $dA_p = dr \cdot d\tilde{A}_p \cdot di_p$, where di_p is the identity map, $d\tilde{A}_p$ maps T_pS^n to $T_{-p}\mathbb{R}^{n+1}$. But we can verify by the observation on T_pS^n that $d\tilde{A}_p$ maps T_pS^n to $T_{-p}S^n$. Thus $dr : T_{-p}\mathbb{R}^{n+1} \longrightarrow T_{-p}S^n$ does not have actual effect in dA_p. Since di_p is identity this gives us that $dA_p = d\tilde{A}_p|_{T_pS^n}$. Hence dA_p is given by $v \mapsto -v$.

• Now we can show that $A : S^n \to S^n$ is an isometry of S^n. Let $i : S^n \hookrightarrow \mathbb{R}^{n+1}$ denote the natural inclusion. Then $g_p(u,v) = \langle di_p(u), di_p(v) \rangle$, and hence

$$g_{-p}(dA_p(u), dA_p(v)) = \langle di_{-p} dA_p(u), di_{-p} dA_p(v) \rangle = \langle -u, -v \rangle = \langle u, v \rangle$$

$$= \langle di_p(u), di_p(v) \rangle = g_p(u,v)$$

which shows that $A : S^n \longrightarrow S^n$ is an isometry.

2°. Denote $\pi : S^n \longrightarrow \mathbb{RP}^n$, $p \longmapsto [p] = \{-p, p\}$ for the double cover of S^n on $\mathbb{RP}^n$.

We know from Chapter 0 that for any $[p] \in \mathbb{RP}^n$ there exists $V_1, V_2 \subset S^n$ open such that $V_1 \cap V_2 = \phi$, $\pi^{-1}(U) = V_1 \cup V_2$, and $\pi|_{V_1} : V_1 \to U$ and $\pi|_{V_2} : V_2 \to U_2$ are both diffeomorphisms. Hence for any $p \in S^n$, the differential map

$$d\pi_p : T_pS^n \longrightarrow T_{[p]}\mathbb{RP}^n$$

is bijective. This allows us to define a Riemannian metric on $\mathbb{RP}^n$ by

$$\langle u, v \rangle_{[p]} := \langle (d\pi_p)^{-1} u, (d\pi_p)^{-1} v \rangle_p \quad \text{for an arbitrary } p \in \pi^{-1}([p]).$$

We need to verify two things. First, the definition of $\langle\,,\rangle_{[p]}$ does not depend on the choice of $p \in \pi^{-1}([p])$. This is because if we choose $-p = A(p)$ instead of p, then $\pi \circ A = \pi \Rightarrow (d\pi_{A(p)}) \cdot dA_p = d\pi_p \Rightarrow (d\pi_p)^{-1} = (dA_p)^{-1} \cdot (d\pi_{A(p)})^{-1}$, and since A is an isometry of S^n, we have

$$\langle (d\pi_p)^{-1} u, (d\pi_p)^{-1} v \rangle_p = \langle (dA_p)^{-1} (d\pi_{A(p)})^{-1} u, (dA_p)^{-1} (d\pi_{A(p)})^{-1} v \rangle_p$$

$$= \langle (dA_p)(dA_p)^{-1} (d\pi_{A(p)})^{-1} u, (dA_p)(dA_p)^{-1} (d\pi_{A(p)})^{-1} v \rangle_{A(p)}$$

$$= \langle (d\pi_{A(p)})^{-1} u, (d\pi_{A(p)})^{-1} v \rangle_{A(p)}$$

which proves that the definition does not depend on the choice of $p \in [p]$.

Second we need to show that $\langle\,,\rangle_{[p]}$ is actually a Riemannian metric on $\mathbb{RP}^n$. Since $\pi_1 := \pi|_{V_1} : V_1 \longrightarrow U$ is a diffeomorphism, so is $\pi_1^{-1} : U \longrightarrow V_1$

Note that $(d\pi_p)^{-1} = d(\pi_i^{-1})_{[p]}$, thus $\langle u, v\rangle_{[p]} = \langle (d\pi_p)^{-1}u, (d\pi_p)^{-1}v\rangle_p =$
$= \langle d(\pi_i^{-1})_{[p]}u, d(\pi_i^{-1})_{[p]}v\rangle_p$. Thus $\langle u,v\rangle_{[p]}$ is the metric induced by $\pi_i^{-1}: \mathcal{U} \to V_i$ (c.f. P39 Example 2.5). Thus $\langle , \rangle_{[p]}$ is indeed a Riemannian metric.

3°. To show that $\pi: S^n \longrightarrow \mathbb{RP}^n$ is a local isometry, for each $p \in S^n$, let V be a neighborhood around p such that $\pi|_V: V \longrightarrow \mathcal{U} := \pi(V)$ is a diffeomorphism. Then for any $u, v \in T_pS^n$ we have that

$$\langle u, v\rangle_p = \langle (d\pi_p)^{-1}(d\pi_p)u, (d\pi_p)^{-1}(d\pi_p)v\rangle_p = \langle (d\pi_p)u, (d\pi_p)v\rangle_{[p]}$$

which implies that $\pi: S^n \longrightarrow \mathbb{RP}^n$ is a local isometry.

2. Introduce a Riemannian metric on the Torus T^n in such a way that the natural projection $\pi: \mathbb{R}^n \longrightarrow T^n$ given by

$$\pi(x_1, \cdots, x_n) = (e^{ix_1}, \cdots, e^{ix_n}), \quad (x_1, \cdots, x_n) \in \mathbb{R}^n,$$

is a local isometry. Show that with this metric T^n is isometric to the flat torus.

• Proof: 1°. First we characterize the tangent space $T_{[p]}T^n$ at some $[p] = (e^{ip_1}, \cdots, e^{ip_n}) \in T^n$. Note that the curve $\gamma_1: (-\varepsilon, \varepsilon) \longrightarrow T^n: t \longmapsto (e^{i(p_1+t)}, e^{ip_2}, \cdots, e^{ip_n})$ lies in T^n, thus it gives rise to a tangent vector $\gamma_1'(0)$ defined by

$$\gamma_1'(0)f = \frac{d}{dt}\Big|_{t=0}(f\circ\gamma_1)(t) = \frac{d}{dt}\Big|_{t=0} f(e^{i(p_1+t)}, e^{ip_2}, \cdots, e^{ip_n}) = \frac{\partial f}{\partial x_1}\Big|_p \cdot ie^{ip_1} = \left(ie^{ip_1}\frac{\partial}{\partial x_1}\Big|_p\right)f$$

Hence $\partial_1 := (ie^{ip_1}, 0, \cdots, 0)$ is a tangent vector to T^n at $[p]$. Similarly, $\partial_2 := (0, ie^{ip_2}, 0, \cdots, 0)$, $\partial_3 := (0, 0, ie^{ip_3}, 0, \cdots, 0)$, $\cdots$, $\partial_n := (0, \cdots, 0, ie^{ip_n})$ are tangent vectors to T^n at $[p]$. Obviously they are linearly independent. Since $T_{[p]}T^n$ is of dimension n, we know $\partial_1, \cdots, \partial_n$ span $T_{[p]}T^n$.

2° Note that $\pi: \mathbb{R}^n \to T^n$ is a smooth covering, and the Euclidean metric on $\mathbb{R}^n$ is invariant under all deck transformations. Thus the induced metric on T^n is well-defined, and $\pi: \mathbb{R}^n \to T^n$ is a local isometry (This can be proved in an exactly identical manner as we did in Problem 1.) To see what this induced metric is, we let $v = (v_1, \cdots, v_n)$ be any tangent vector to $\mathbb{R}^n$ at $p = (p_1, \cdots, p_n)$. Then $\gamma(t) = (p_1 + tv_1, \cdots, p_n + tv_n)$ is a curve in $\mathbb{R}^n$, and thus $(\pi\circ\gamma)(t) = (e^{i(p_1+tv_1)}, \cdots, e^{i(p_n+tv_n)})$ is a curve on T^n. It is easy to verify that $d\pi_p(v) = (iv_1e^{ip_1}, \cdots, iv_ne^{ip_n})$. Since $d\pi_p$ is bijective, we know from this expression that $(d\pi_p)^{-1}\partial_j = (0, \cdots, 0, \underset{j\text{-th entry}}{1}, 0, \cdots, 0)$.
Thus the induced metric on T^n by $\pi: \mathbb{R}^n \to T^n$ is determined by

$$\langle \partial_j, \partial_k\rangle_\pi = \langle (d\pi_p)^{-1}\partial_j, (d\pi_p)^{-1}\partial_k\rangle = \langle e_j, e_k\rangle = \delta_{jk}.$$

No.

Date

3°. We now determine the product metric on T^n. It is direct verify that the differential of the natural projection $\pi_j : T^n = S_1^1 \times \cdots \times S_n^1 \longrightarrow S_j$ at p is simply the map which takes the j-th coordinate of the tangent vector in $T_{[p]}T^n$. Thus for any $\partial_j, \partial_k \in T_{[p]}T^n$ such that $j \neq k$ we must have $\langle \partial_j, \partial_k \rangle_{Prod} = \sum_{m=1}^{n} \langle d\pi_{m,p}(\partial_j), d\pi_{m,p}(\partial_k) \rangle$ and the inner product of any vector in TS^1 with 0 is just 0. For any $\partial_j \in T_{[p]}T^n$, $\langle \partial_j, \partial_j \rangle_{Prod} = \sum_{m=1}^{n} \langle d\pi_{m,p}(\partial_j), d\pi_{m,p}(\partial_j) \rangle_{Prod} = \langle ie^{iP_j}, ie^{iP_j} \rangle_{S^1}$.
Recall that the canonical metric on S^1 is induced from $\mathbb{R}^2$, and the Euclidean metric on $\mathbb{R}^2$ is equivalent to the Hermitian metric on $\mathbb{C}^1$, thus we have $\langle \partial_j, \partial_j \rangle_{Prod} = \langle ie^{iP_j}, ie^{iP_j} \rangle_{S^1} = ie^{iP_j} \cdot (-i) e^{-iP_j} = i \cdot (-i) = 1$. Hence $\langle \partial_j, \partial_k \rangle_{Prod} = \delta_{jk}$.

4°. Taking the diffeomorphism $f : (T^n, \langle , \rangle_\pi) \longrightarrow (T^n, \langle , \rangle_{Prod})$ to be the identity, and noting that $\langle \partial_j, \partial_k \rangle_\pi = \delta_{jk} = \langle \partial_j, \partial_k \rangle_{Prod}$, we see that the metric on T^n induced by $\pi : \mathbb{R}^n \to T^n$ is isometric to the flat torus.

3. Obtain an isometric immersion of the flat torus T^n into $\mathbb{R}^{2n}$.

- Solution: We parametrize T^n by $T^n := \{(e^{i\theta_1}, \cdots, e^{i\theta_n}) \mid 0 \le \theta_j < 2\pi, 1 \le j \le n\}$.
Define $f : T^n \to \mathbb{R}^{2n}$ by $(e^{i\theta_1}, \cdots, e^{i\theta_n}) \longmapsto (\cos\theta_1, \sin\theta_1, \cos\theta_2, \sin\theta_2, \cdots, \cos\theta_n, \sin\theta_n)$.
First we compute df. At each $[p] \in T^n$, where $[p] = (e^{iP_1}, \cdots, e^{iP_n})$, we have $T_{[p]}T^n = \mathrm{span}\{\partial_1, \cdots, \partial_n\}$, $\partial_j = (0, \cdots, 0, ie^{iP_j}, 0, \cdots, 0)$. This is what we have already done in Problem 2. Then it is easy to verify that (by constructing a curve like $(\pi \circ \gamma)(t)$ in Problem 2 and take derivatives with respect to t explicitly)

$$df_{[p]}(\partial_j) = (0, 0, \cdots, 0, 0, \underset{(2j-1)\text{-th entry}}{-\sin P_j}, \underset{(2j)\text{-th entry}}{\cos P_j}, 0, 0, \cdots, 0, 0) =: \tilde{\partial}_j.$$

Now it is obvious to see that $df_{[p]}$ is injective for each $[p] \in T^n$ (here we assumed that $0 < P_j < 2\pi$, $1 \le j \le n$, i.e. we restricted $[P]$ to one of the two charts on T^n. By an identical argument we can show all statements hold on the other chart.) because $df_{[p]}(v) = 0$ must imply $v = 0$. (We can show that $\{\tilde{\partial}_j\}_{j=1}^n$ is a linearly independent set.) Moreover, since $\langle \partial_j, \partial_k \rangle_{Prod} = \delta_{jk} = \langle \tilde{\partial}_j, \tilde{\partial}_k \rangle_{\mathbb{R}^{2n}} = \langle df_{[p]}(\partial_j), df_{[p]}(\partial_k) \rangle_{\mathbb{R}^{2n}}$ we know f is an isometry. Thus $f : T^n \to \mathbb{R}^{2n}$ is an isometric immersion.

$$(e^{i\theta_1}, \cdots, e^{i\theta_n}) \mapsto (\cos\theta_1, \sin\theta_1, \cos\theta_2, \sin\theta_2, \cdots, \cos\theta_n, \sin\theta_n)$$

Remark. See P.39 Example 2.5 for the definition of an isometric immersion. Here we constructed an immersion from the flat torus to $\mathbb{R}^{2n}$ such that the induced metric on $\mathbb{R}^{2n}$ is exactly the canonical Euclidean metric on $\mathbb{R}^{2n}$.

4. A function $g:\mathbb{R}\to\mathbb{R}$ given by $g(t)=yt+x$, $t,x,y\in\mathbb{R}$, $y>0$, is called a proper affine function. The subset of all such functions with respect to the usual composition law forms a Lie group G. As a differentiable manifold G is simply the upper half-plane $\{(x,y)\in\mathbb{R}^2;\ y>0\}$ with the differentiable structure induced from $\mathbb{R}^2$. Prove that:

(a) The left-invariant Riemannian metric of G which at the neutral element $e=(0,1)$ coincides with the Euclidean metric ($g_{11}=g_{22}=1$, $g_{12}=0$) is given by $g_{11}=g_{22}=\frac{1}{y^2}$, $g_{12}=0$ (this is the metric of the non-euclidean geometry of Lobatchevski).

(b) Putting $(x,y)=z=x+iy$, $i=\sqrt{-1}$, the transformation $z\mapsto z'=\frac{az+b}{cz+d}$, $a,b,c,d\in\mathbb{R}$, $ad-bc=1$ is an isometry of G.

Hint: Observe that the first fundamental form can be written as:

$$ds^2=\frac{dx^2+dy^2}{y^2}=-\frac{4\,dz\,d\bar{z}}{(z-\bar{z})^2}.$$

- Proof: (a) First we verify that G is a Lie group. For any $g(t)=yt+x\in G$, $g^{-1}(t)=\frac{1}{y}t-\frac{x}{y}$. In local coordinates this gives $(x,y)\xmapsto{\text{inv.}}(-\frac{x}{y},\frac{1}{y})$, which is obviously a differentiable map from $\mathbb{R}^2_+$ to $\mathbb{R}^2_+$. For any $g_1(t)=y_1t+x_1\in G$, $g_2(t)=y_2t+x_2\in G$, $(g_2g_1)(t)=y_2(y_1t+x_1)+x_2=(y_2y_1)t+(y_2x_1+x_2)$. In coordinates this gives $((x_2,y_2),(x_1,y_1))\xmapsto{\text{mult.}}(y_2x_1+x_2,\ y_2y_1)$, which is obviously a differentiable map from $\mathbb{R}^2_+$ to $\mathbb{R}^2_+$. Thus G is actually a Lie group.

Now we fix a point $p\in G$ with coord. (x,y). Let $\gamma_1(s)$ be a curve in $\mathbb{R}^2_+$ given as

$$\gamma_1(s):(-\varepsilon,\varepsilon)\longrightarrow\mathbb{R}^2_+$$
$$s\longmapsto(x+s,y)$$

Then $\gamma_1'(0)=(1,0)=\partial/\partial x\in T_{(x,y)}\mathbb{R}^2_+$, and $dg_{(x,y)}(\partial/\partial x)=\partial_1\in T_pG$. Here we denote g for the coordinate map from $\mathbb{R}^2_+$ to G. Thus in coordinates

$$(dg^{-1})_e(dL_{p^{-1}})_p(\partial_1)=(dg^{-1})_e(dL_{p^{-1}})_p(dg_{(x,y)})(\partial/\partial x)=d(g^{-1}\circ L_{p^{-1}}\circ g)_{(x,y)}(\partial/\partial x)$$
$$=\frac{d}{ds}\Big|_{s=0}(g^{-1}\circ L_{p^{-1}}\circ g)(\gamma_1(s))=\frac{d}{ds}\Big|_{s=0}\left(\frac{s}{y},1\right)=\left(\frac{1}{y},0\right)$$

Similarly, let $\gamma_2(s)$ be another curve in $\mathbb{R}^2_+$ given as

$$\gamma_2(s):(-\varepsilon,\varepsilon)\longrightarrow\mathbb{R}^2_+ \qquad |\varepsilon|<|y|$$
$$s\longmapsto(x,y+s)$$

Then $\gamma_2'(0)=(0,1)=\partial/\partial y\in T_{(x,y)}\mathbb{R}^2_+$, and $dg_{(x,y)}(\partial/\partial y)=\partial_2\in T_pG$. Thus

$$(dg^{-1})_e\,(dL_{p^{-1}})_p\,(\partial_2) = (dg_e^{-1})(dL_{p^{-1}})_p(dg_{(x,y)})(\partial/\partial y) = d(g^{-1}\circ L_{p^{-1}}\circ g)_{(x,y)}(\partial/\partial y)$$

$$= \frac{d}{ds}\Big|_{s=0}(g^{-1}\circ L_{p^{-1}}\circ g)(\gamma_2(s)) = \frac{d}{ds}\Big|_{s=0}\left(0,\ 1+\frac{s}{y}\right) = \left(0, \frac{1}{y}\right)$$

Therefore:

$$\langle \partial_1, \partial_1\rangle_p = \langle (dL_{p^{-1}})_p(\partial_1),\ (dL_{p^{-1}})_p(\partial_1)\rangle_e = \langle dg_{(x,y)}((\tfrac{1}{y},0)),\ dg_{(x,y)}((0,\tfrac{1}{y}))\rangle_e$$

$$= \frac{1}{y^2}\langle dg_{(x,y)}((1,0)),\ dg_{(x,y)}((1,0))\rangle = \frac{1}{y^2}\cdot 1 = \frac{1}{y^2}.$$

$$\langle \partial_2, \partial_2\rangle_p = \langle (dL_{p^{-1}})_p(\partial_2),\ (dL_{p^{-1}})_p(\partial_2)\rangle_e = \langle dg_{(x,y)}((0,\tfrac{1}{y})),\ dg_{(x,y)}((0,\tfrac{1}{y}))\rangle_e$$

$$= \frac{1}{y^2}\langle dg_{(x,y)}((0,1)),\ dg_{(x,y)}((0,1))\rangle_e = \frac{1}{y^2}\cdot 1 = \frac{1}{y^2}.$$

$$\langle \partial_1, \partial_2\rangle_p = \langle (dL_{p^{-1}})_p(\partial_1),\ (dL_{p^{-1}})_p(\partial_2)\rangle_e = \langle dg_{(x,y)}((\tfrac{1}{y},0)),\ dg_{(x,y)}((0,\tfrac{1}{y}))\rangle_e$$

$$= \frac{1}{y^2}\langle dg_{(x,y)}((1,0)),\ dg_{(x,y)}((0,1))\rangle_e = \frac{1}{y^2}\cdot 0 = 0.$$

(b) First we show that $M: z \mapsto w = \frac{az+b}{cz+d}$ is well-defined as a map from G to G. Indeed,

$$w = \frac{az+b}{cz+d} = \frac{(az+b)(c\bar{z}+d)}{|cz+d|^2} = \frac{((ax+b)+iay)((cx+d)-icy)}{|cz+d|^2}$$

$$\Rightarrow \operatorname{Im} w = \frac{ay(cx+d) - cy(ax+b)}{|cz+d|^2} = \frac{\operatorname{Im} z}{|cz+d|^2} > 0$$

thus $M: z \mapsto w = \frac{az+b}{cz+d}$ is a well-defined map from G to G.

Direct computation gives

$$M^* ds^2 = -\frac{dw\, d\bar{w}}{(w-\bar{w})^2} = -\frac{d\left(\frac{az+b}{cz+d}\right) d\left(\frac{a\bar{z}+b}{c\bar{z}+d}\right)}{\left(\frac{az+b}{cz+d} - \frac{a\bar{z}+b}{c\bar{z}+d}\right)^2} = -\frac{\left[\frac{a(cz+d)-c(az+b)}{(cz+d)^2}dz\right]\left[\frac{a(c\bar{z}+d)-c(a\bar{z}+b)}{(c\bar{z}+d)^2}d\bar{z}\right]}{\frac{[(az+b)(c\bar{z}+d)-(cz+d)(a\bar{z}+b)]^2}{(cz+d)^2(c\bar{z}+d)^2}}$$

$$= -\frac{(ad-bc)^2\, dz\, d\bar{z}}{(ad-bc)^2\,(z-\bar{z})^2} = -\frac{dz\,d\bar{z}}{(z-\bar{z})^2} = ds^2.$$ where we only used $ad-bc \neq 0$.

Hence M is an isometry of G.

Remark. Here is a faster and more concise approach to (a): At $g_0 = (x_0, y_0)$, consider L_{g_0}. Assume $g(t) = yt+x$, then $(L_{g_0}(g))(t) = (g_0\circ g)(t) = y_0(yt+x)+x_0 = (y_0 y)t + (y_0 x + x_0)$.

$$L_{g_0}: G \longrightarrow G$$
$$e = (0,1) \longmapsto g_0$$
$$(x,y) \longmapsto (y_0 x + x_0,\ y_0 y)$$

Thus $dL_{g_0}: T_eG \longrightarrow T_{g_0}G$ i.e. if $u = (a,b) = a\frac{\partial}{\partial x} + b\frac{\partial}{\partial y} \in T_eG$

$(a,b) \longmapsto (y_0 a, y_0 b)$ then $(dL_{g_0})(u) = y_0 u$.

Denote $\langle \cdot,\cdot \rangle$ for the left-invariant Riemannian metric of G which coincides with the canonical Euclidean metric at $e=(0,1)$. Then $\forall\, u, v \in T_{g_0}G$

$$\langle u, v\rangle_{g_0} = \left\langle (dL_{g_0})^{-1}u,\ (dL_{g_0})^{-1}v \right\rangle_e = \langle y_0^{-1}u,\ y_0^{-1}v\rangle_e = \frac{1}{y_0^2}\langle u, v\rangle_e.$$

Thus $g_{11} = g_{22} = \frac{1}{y_0^2}$, $g_{12} = g_{21} = 0$ because $\langle \partial_j, \partial_k\rangle_e = \delta_{jk}$.

5. Prove that the isometries of $S^n \subset \mathbb{R}^{n+1}$, with the induced metric, are the restrictions to S^n of the linear orthogonal maps of $\mathbb{R}^{n+1}$.

- Proof: We first show the following lemma:

LEMMA: Let (M, g_M) and (N, g_N) be two Riemannian manifolds of dimension n. Assume M is connected, and $f_i: M \longrightarrow f_i(M) \subset N$, $i = 1,2$ are two local isometries onto their images. If there exists $p \in M$ such that $f_1(p) = f_2(p)$ and $(df_1)_p = (df_2)_p$, then $f_1 = f_2$ on M.

Proof of the LEMMA Define $A := \{x \in M \mid f_1(x) = f_2(x) \text{ and } (df_1)_x = (df_2)_x\}$. Then $A \neq \phi$ because $p \in A$. A is also closed since f_1, f_2 are both smooth. We want to show that A is also open. To see this, take any $q \in A$. Since $\dim M = \dim N = n$, $f_1(M)$ and $f_2(M)$ are both open in N (due to invariance of domains). Thus we can find open neighborhoods V for $f_1(q) = f_2(q)$ in N, and two open neighborhoods $\mathcal{U}_1, \mathcal{U}_2$ of q in M such that $f_i|_{\mathcal{U}_i}: \mathcal{U}_i \longrightarrow V$ are both isometries (for $i=1,2$). Pick a normal neighborhood $\mathcal{U} \subset \mathcal{U}_1 \cap \mathcal{U}_2$, and consequently the exponential map $\exp: T_qM \supset W \longrightarrow \mathcal{U}$ is a diffeomorphism for some open neighborhood W. Define $\phi := (f_2|_{\mathcal{U}_2})^{-1} \circ (f_1|_{\mathcal{U}_1}) : \mathcal{U}_1 \cap \mathcal{U}_2 \longrightarrow \mathcal{U}_1 \cap \mathcal{U}_2$, then ϕ is an isometry on $\mathcal{U}_1 \cap \mathcal{U}_2$ with $\phi(q) = q$ and $(d\phi)_q = \mathrm{Id}$. Now let γ be a geodesic arc in $\mathcal{U}$ starting at $q \in \mathcal{U} \subset \mathcal{U}_1 \cap \mathcal{U}_2$, then $\phi \circ \gamma$ is also a geodesic arc starting at q. Note that $\phi\circ\gamma(0) = \phi(q) = q = \gamma(0)$, and the tangent vectors $(\phi\circ\gamma)'(0) = (d\phi)_q(\gamma'(0)) = (\mathrm{Id})(\gamma'(0)) = \gamma'(0)$ are identical. Hence $\phi\circ\gamma = \gamma$ by the uniqueness of the solution to linear ODE systems. It follows that $\phi \circ \exp = \exp$, which implies $\phi|_{\mathcal{U}} = (\mathrm{id})|_{\mathcal{U}}$ because $\exp: W \longrightarrow \mathcal{U}$ is a diffeomorphism from W onto $\mathcal{U}$. By the definition of ϕ, this implies that $f_1|_{\mathcal{U}} = f_2|_{\mathcal{U}}$ and $(df_1)|_{\mathcal{U}} = (df_2)|_{\mathcal{U}}$. Thus $\mathcal{U} \subset A$ and A is open. Since A is connected, this gives $A = M$. ∎

No.

Date

Now we can show that $\mathrm{Isom}(S^n) = O(n+1)|_{S^n}$.

On the one hand, $\mathrm{Isom}(S^n) \supset O(n+1)|_{S^n}$ because for any $A \in O(n+1)$ and any $x \in S^n$ we have $\langle Ax, Ax\rangle_{\mathbb{R}^{n+1}} = \langle x, x\rangle_{\mathbb{R}^{n+1}} = 1$. Hence A maps S^n to S^n. Moreover, since S^n has the induced metric from $\mathbb{R}^{n+1}$, we have for any $v, w \in T_xS^n$:

$g(dA_x(v), dA_x(w)) = \langle dA_x(v), dA_x(w)\rangle_{\mathbb{R}^{n+1}} = \langle Av, Aw\rangle_{\mathbb{R}^{n+1}} = \langle v, w\rangle_{\mathbb{R}^{n+1}} = g(v,w)$

where we applied the linearity of $A \in O(n+1)$ to conclude that $dA_x = A$.

Thus $dA_x = A|_{T_xS^n} \in \mathrm{Isom}(S^n)$, i.e. $O(n+1)|_{S^n} \subset \mathrm{Isom}(S^n)$.

On the other hand, any $f \in \mathrm{Isom}(S^n)$ extends to an isometry on $\mathbb{R}^{n+1}$, thus the extended map is a linear orthogonal map, which then gives $f \in O(n+1)|_{S^n}$.

To see this, recall our observation at the beginning of the solution to Problem 1 in Chapter 1. It tells us that for any $x \in S^n \subset \mathbb{R}^{n+1}$ we have $\mathbb{R}^{n+1} = \mathbb{R}x \oplus T_xS^n$. Since $1 = \langle x, x\rangle_{\mathbb{R}^{n+1}} = \langle f(x), f(x)\rangle_{\mathbb{R}^{n+1}}$ because $f(x) \in S^n$ and $g(v,w) = g(df_x(v), df_x(w))$ because $f \in \mathrm{Isom}(S^n)$, we can define a linear map by

$$A : \mathbb{R}^{n+1} = \mathbb{R}x \oplus T_xS^n \longrightarrow \mathbb{R}^{n+1}$$
$$(\lambda x, v) \longmapsto \lambda f(x) + df_x(v)$$

for any $\lambda \in \mathbb{R}$, $v \in T_xS^n$.

Then $A : \mathbb{R}^{n+1} \to \mathbb{R}^{n+1}$ is an isometry because

$$\begin{aligned}
\langle A(\lambda_1 x, v), A(\lambda_2 x, w)\rangle_{\mathbb{R}^{n+1}} &= \langle \lambda_1 f(x) + df_x(v), \lambda_2 f(x) + df_x(w)\rangle_{\mathbb{R}^{n+1}} \\
&= \langle \lambda_1 f(x), \lambda_2 f(x)\rangle_{\mathbb{R}^{n+1}} + \langle df_x(v), df_x(w)\rangle_{\mathbb{R}^{n+1}} \quad \text{(orthogonality of subspaces)} \\
&= \langle \lambda_1 x, \lambda_2 x\rangle_{\mathbb{R}^{n+1}} + \langle v, w\rangle_{\mathbb{R}^{n+1}} \quad (f \in \mathrm{Isom}(S^n)) \\
&= \langle (\lambda_1 x, v), (\lambda_2 x, w)\rangle_{\mathbb{R}^{n+1}}.
\end{aligned}$$

If we restrict A to S^n, then $A(x,0) = f(x)$ for all $x \in S^n$, and $dA_x|_{T_xS^n} = A|_{T_xS^n} = df_x|_{T_xS^n}$. Now it follows from the lemma proved above that $f = A|_{S^n}$. (That $A|_{S^n}$ is an isometry on S^n follows from the fact that the canonical metric defined on S^n is induced from $\langle\, ,\rangle_{\mathbb{R}^{n+1}}$.) Thus $\mathrm{Isom}(S^n) \subset O(n+1)|_{S^n}$ which combined with the "on the one hand" part gives us $\mathrm{Isom}(S^n) = O(n+1)|_{S^n}$.

<u>Remark</u>: An alternative way to say that $O(n+1) = \mathrm{Isom}(\mathbb{R}^{n+1})$ is as follows: given any $p, q \in S^n$ any any orthonormal bases $\{v_1, \dots, v_n\} \subset T_pS^n$ and $\{w_1, \dots, w_n\} \subset T_qS^n$, there exists a linear orthonormal map taking p to q and $\{v_1, \dots, v_n\}$ to $\{w_1, \dots, w_n\}$. To see this, first assume $p = q =$ north pole and $\{v_1, \dots, v_n\} = \{e_2, \dots, e_{n+1}\}$ where $\{e_1, \dots, e_{n+1}\}$ is the canonical orthonormal basis of $\mathbb{R}^{n+1}$. Then given $\{w_1, \dots, w_n\}$ we can construct an orthonormal matrix explicitly. The general case follows immediately through "ferry through the north pole".

6. Show that the relation "M is locally isometric to N" is not a symmetric relation.

- Proof: Consider the covering map $f: \mathbb{R} \to S^1$. Equip S^1 with the canonical metric induced from $\mathbb{R}^2$, and equip $\mathbb{R}$ with the "covering map" induced from the canonical metric on S^1. Then $f: \mathbb{R} \to S^1$ is a local isometry, i.e. $\mathbb{R}$ is locally isometric to S^1. If there exists a local isometry $g: S^1 \to \mathbb{R}$, then g achieves its minimum and maximum on S^1 since S^1 is compact. Thus $(dg)_p$ for some $p \in S^1$, and hence g is not a local isometry in any neighborhood of p. Thus S^1 is not isometric to $\mathbb{R}$.

Remark: For any curve $\gamma: (-\varepsilon, \varepsilon) \to S^1$ with $\gamma(0) = p$, we can get $g \circ \gamma: (-\varepsilon, \varepsilon) \to \mathbb{R}$ with $(g \circ \gamma)'(0) = (dg_p)(\gamma'(0)) \in \mathbb{R}$.

Remark: By definition, if M is locally isometric to N then there exists a local isometry f from M to N, and f has to be globally defined on M instead of possibly only defined in a neighborhood around some point in M, by the definition of a "local isometry" (cf. P.39 Definition 2.3).

7. Let G be a compact connected Lie group ($\dim G = n$). The object of this exercise is to prove that G has a bi-invariant Riemannian metric. To do this, take the following approach:

(a) Let ω be a differential n-form on G invariant on the left, that is, $L_x^* w = \omega$ for all $x \in G$. Prove that ω is right invariant.

Hint: For any $a \in G$, $R_a^* \omega$ is left invariant. It follows that $R_a^* \omega = f(a)\, \omega$. Verify that $f(ab) = f(a) f(b)$, that is, $f: G \to \mathbb{R} - \{0\}$ is a (continuous) homomorphism of G into the multiplicative group of real numbers. Since $f(G)$ is a compact connected subgroup, the conclusion $f(G) = \{1\}$ holds. Therefore $R_a^* \omega = \omega$.

(b) Show that there exists a left invariant differential n-form ω on G.

(c) Let $\langle , \rangle$ be a left invariant metric on G. Let ω be a positive differential n-form on G which is invariant on the left, and define a new Riemannian metric $\langle\langle , \rangle\rangle$ on G by $\langle\langle u, v \rangle\rangle_y = \int_G \langle (dR_x)_y u, (dR_x)_y v \rangle_{yx}\, \omega$, Prove that this new Riemannian metric $\langle\langle , \rangle\rangle$ is ~~bi-invariant.~~ $x, y \in G$, $u, v \in T_y(G)$.

No.

Date

- Proof: (a) For any $a \in G$, $L_b^*(R_a^*\omega) = (R_a \circ L_b)^*\omega = (L_b \circ R_a)^*\omega = R_a^*(L_b^*\omega) = R_a^*\omega$, thus $R_a^*\omega$ is left invariant. At the identity element $e \in G$, since $\Lambda^n T_e^* G$ is one-dimensional, there exists a unique real number $f(a)$ such that $(R_a^*\omega)_e = f(a)\omega_e$. Since R_a is differentiable (smooth), $f(a)$ depends smoothly on $a \in G$. By left invariance we have $(R_a^*\omega)_p = L_{p^{-1}}^*(R_a^*\omega)_e = L_{p^{-1}}^*(f(a)\omega_e) = f(a) L_{p^{-1}}^*(\omega_e) = f(a)\omega_p$ at any $p \in G$. Thus $R_a^*\omega = f(a)\omega$ for some function f smoothly depending on $a \in G$. Moreover, since $R_a^* \circ R_b^* = (R_b \circ R_a)^* = R_{ab}^*$, we know that
$f(ab)\,\omega = R_{ab}^*\omega = (R_a^* \circ R_b^*)\omega = f(a) R_b^*\omega = f(a)f(b)\,\omega$.
If ω is globally vanishing, then there is nothing to be shown. If ω is nonzero at some point $p \in G$, then $R_a^*\omega$ is also nonzero at p because R_a is a diffeomorphism and thus $dR_a : T_{pa^{-1}}G \longrightarrow T_p G$ is bijective. In this case $f(a) \neq 0$. Thus f is a smooth homomorphism from G to $\mathbb{R}\setminus\{0\}$, the multiplicative group of real numbers. Note that G is compact and connected, $f(G)$ is a compact connected subgroup of $\mathbb{R}\setminus\{0\}$. If a subgroup of $\mathbb{R}\setminus\{0\}$ contains an element $k \neq 0$ such that $|k| \neq 1$, then it also contains k^n and k^{-n} for any $n \in \mathbb{N}$, and hence is not bounded and thus non-compact. Moreover, by the connectedness we know $f(G)$ can not be $\{1, -1\}$, thus $f(G) = \{1\}$. This tells us that $R_a^*\omega = \omega$ for any $a \in G$, i.e. ω is also right invariant.

(b) Pick an arbitrary non-zero n-form ω_e in $\Lambda^n T_e^* G$, and define an n-form at $\Lambda^n T_p^* G$ by $\omega_p := L_{p^{-1}}^*\omega_e$. Then $L_g^*\omega_p = L_g^* L_{p^{-1}}^*\omega_e = L_{p^{-1}g}^*\omega_e = \omega_{g^{-1}p}$, $\forall g \in G$, which tells us $L_g^*\omega = \omega$, i.e. ω is left invariant. Note that $L_{p^{-1}}$ is a diffeomorphism for any $p \in G$, ω is a globally non-vanishing n-form on G. This proves the existence of a left-invariant n-form on G. In particular, since ω is no-where vanishing on G, we see that G is orientable.

(c) For any fixed $y \in G$, $u, v \in T_y G$, let $f_y(x) := \langle (dR_x)_y u, (dR_x)_y v \rangle_{yx}$. Then $\langle\langle u, v \rangle\rangle_y = \int_G f_y(x)\,\omega$. The integral exists because G is compact. Since ω is positive, $f_y(x)$ comes from a positively definite inner product, we know $\langle\langle\, ,\, \rangle\rangle$ is also positively definite. That $\langle\langle\, ,\, \rangle\rangle$ is symmetric follows from the symmetry of $\langle\, ,\, \rangle$. Thus $\langle\langle\, ,\, \rangle\rangle$ is a well-defined Riemannian metric on G. (That $\langle\langle\, ,\, \rangle\rangle_y$ depends smoothly on $y \in G$ follows from the smoothness of R_x, $\langle\, ,\, \rangle_{yx}$ and the integral with respect to $y \in G$. The integral depends smoothly on $y \in G$ because G is compact and thus $f(x)$ has some "uniform smoothness" on G.)

First we show that $\langle\langle\, , \rangle\rangle$ is left-invariant. Basically this follows from the left-invariance of the metric $\langle\, , \rangle$. Indeed, for any $b \in G$ and any $u, v \in T_y G$

$$\langle\langle dL_b(u), dL_b(v)\rangle\rangle_{by} = \int_G \langle (dR_x)_{by}(dL_b)_y(u), (dR_x)_{by}(dL_b)_y(v)\rangle_{byx}\, \omega$$

$$= \int_G \langle d(R_x \circ L_b)_y(u), d(R_x \circ L_b)_y(v)\rangle_{byx}\, \omega$$

$$= \int_G \langle d(L_b \circ R_x)_y(u), d(L_b \circ R_x)_y(v)\rangle_{byx}\, \omega$$

$$= \int_G \langle (dL_b)_{yx}(dR_x)_y(u), (dL_b)_{yx}(dR_x)_y(v)\rangle_{byx}\, \omega$$

$$= \int_G \langle (dR_x)_y(u), (dR_x)_y(v)\rangle_{yx}\, \omega = \langle\langle u, v\rangle\rangle_y .$$

Second we show that $\langle\langle\, , \rangle\rangle$ is right-invariant. To see this, first note that the right-invariance of ω gives $R_b^*\omega = \omega$ for any $b \in G$, thus for any $b \in G$ the right action R_b is an orientation preserving diffeomorphism on G (see GTM98, P43 Definition 5.4). And it follows that $\int_G f_y(x)\,\omega = \int_G R_b^*(f_y(x)\omega)$. (see GTM98, P45 Remark 5.8).
We now have

$$\langle\langle dR_b(u), dR_b(v)\rangle\rangle_{yb} = \int_G \langle (dR_x)_{yb}(dR_b)_y(u), (dR_x)_{yb}(dR_b)_y(v)\rangle_{ybx}\, \omega$$

$$= \int_G \langle d(R_x \circ R_b)_y(u), d(R_x \circ R_b)_y(v)\rangle_{ybx}\, \omega = \int_G \langle (dR_{bx})_y(u), (dR_{bx})_y(v)\rangle_{y(bx)}\, \omega$$

$$= \int_G f_y(bx)\,\omega = \int_G (f_y \circ R_b)(x)\, R_b^*\omega = \int_G R_b^*(f_y(x)\,\omega)$$

$$= \int_G f_y(x)\,\omega = \langle\langle u, v\rangle\rangle_y .$$

Therefore, the Riemannian metric $\langle\langle\, , \rangle\rangle$ on G is bi-invariant. Hence every compact connected Lie group admits a bi-invariant Riemannian metric which is basically "averaging out the metrics induced by the push-forwards of a left-invariant metric through right group multiplications". We can also switch the roles of left and right actions in this construction of bi-invariant Riemannian metric.

No.

Date

1. Let M be a Riemannian manifold. Consider the mapping
$$P = P_{c,t_0,t} : T_{c(t_0)}M \longrightarrow T_{c(t)}M$$
defined by: $P_{c,t_0,t}(v)$, $v \in T_{c(t_0)}M$, is the vector obtained by parallel transporting the vector v along the curve c. Show that P is an isometry and that, if M is oriented, P preserves the orientation.

- Proof: We assume the connection is Levi-Civita.

1°. For any $v_1, v_2 \in T_{c(t_0)}M$, by P_{52} Proposition 2.6 we know there exist parallel vector fields V_1, V_2 along the differentiable curve c such that $V_1(t_0) = v_1$, $V_2(t_0) = v_2$. Since the connection is compatible with the metric, we have $\frac{d}{dt}\langle V_1(t), V_2(t)\rangle_{c(t)} = \left\langle \frac{DV_1}{dt}(t), V_2(t)\right\rangle_{c(t)} + \left\langle V_1(t), \frac{DV_2}{dt}(t)\right\rangle_{c(t)} = 0$ for any t.
Hence $\langle P_{c,t_0,t}(v_1), P_{c,t_0,t}(v_2)\rangle_{c(t)} = \langle V_1(t), V_2(t)\rangle_{c(t)} = \langle V_1(t_0), V_2(t_0)\rangle_{c(t_0)} = \langle v_1, v_2\rangle_{c(t_0)}$, which proves that (in a reluctant sense) P is an isometry.

2°. Assume M is oriented (and connected, of course, c.f. P_{18} the definition of an orientation-preserving map). Let $\{(U_\alpha, x_\alpha)\}$ be an oriented atlas on M. Then for any t, $\{\frac{\partial}{\partial x_j}|_{c(t)}\}_{1\le j\le n}$ is a basis of $T_{c(t)}M$. Under this basis, P has a matrix representation $P = P(t_0, t)$. Since parallel transportation is reversible, $P(t_0,t)$ is nonsingular for any t. Note that $\lim_{t\to t_0} P(t_0,t) = I$, and that $P(t_0,t)$ is continuous with respect to t, we know it always holds that $\det P(t_0,t) > 0$ because $\det P(t_0,t_0) = \det I > 0$ and $P(t_0,t)$ is nonsingular for any t.

Remark 1: An alternative way would be like this: the orientability of M gives rise to a globally non-vanishing form $\Omega \in \Lambda^n T^*M$. Let $\{v_1(0), \cdots, v_n(0)\}$ be an orthonormal basis of $T_{c(t_0)}M$ such that $\Omega(v_1(0), \cdots, v_n(0)) = 1$. The parallel transportations turn orthonormal basis into orthonormal basis, thus $\Omega(v_1(t), \cdots, v_n(t)) \neq 0$ for all t. Then by the continuity (smoothness) of $\Omega(v_1(t), \cdots, v_n(t))$ we know it always holds that $\Omega(v_1(t), \cdots, v_n(t)) > 0$ for any t.

Remark 2: In this proof, where did we use the assumption that M is oriented when proving P is orientation-preserving? Is it only so would the term "orientation-preserving" be well-defined? If so, what can we say about P for non-orientable manifolds? Is there a "local" or "along the curve" version of "orientation-preserving"?

Remark 3: See hand-written comments on P_{56} beside this problem for why this is a stupid problem.

No.

Date

2. Let X and Y be differentiable vector fields on a Riemannian manifold M. Let $p \in M$ and let $c: I \longrightarrow M$ be an integral curve of X through p, i.e. $c(t_0)=p$ and $\frac{dc}{dt} = X(c(t))$. Prove that the Riemannian connection of M is

$$(\nabla_X Y)(p) = \frac{d}{dt}\left(P^{-1}_{c,t_0,t}(Y(c(t)))\right)\Big|_{t=t_0},$$

where $P_{c,t_0,t}: T_{c(t_0)}M \longrightarrow T_{c(t)}M$ is the parallel transport along c, from t_0 to t (this shows how the connection can be reobtained from the concept of parallelism).

- Proof: Choose an orthonormal basis $\{P_1(t_0), \cdots, P_n(t_0)\}$ of $T_{c(t_0)}M$. Using Proposition 2.6, we can extend the vectors $P_i(t_0)$, $i=1,\cdots,n$, along c by parallel translation. Because ∇ is compatible with the metric, $\{P_1(t), \cdots, P_n(t)\}$ is an orthonormal basis of $T_{c(t)}M$. Assume under this basis we have $Y(c(t)) = y_1(c(t))P_1(t) + \cdots + y_n(c(t))P_n(t)$. Then $P^{-1}_{c,t_0,t}(Y(c(t))) = y_1(c(t))P_1(t_0) + \cdots + y_n(c(t))P_n(t_0)$ and we have

$$\frac{d}{dt}\left(P^{-1}_{c,t_0,t}(Y(c(t)))\right)\Big|_{t=t_0} = \frac{d}{dt}(y_1 \circ c)(t)\Big|_{t=t_0} P_1(t_0) + \cdots + \frac{d}{dt}(y_n \circ c)(t)\Big|_{t=t_0} P_n(t)$$

$$= X(y_1)(p)P_1(t_0) + \cdots + X(y_n)(p)P_n(t).$$

On the other hand, we have

$$(\nabla_X Y)(p) = \nabla_X(y_1P_1 + \cdots + y_nP_n)(p) = \sum_{k=1}^{n} X(y_k)(p)P_k(t_0) + \sum_{k=1}^{n} y_k(c(t_0))\nabla_X P_k(p)$$

$$= X(y_1)(p)P_1(t_0) + \cdots + X(y_n)(p)P_n(t_0)$$ because $\nabla_X P_k \equiv 0$ for all $1 \le k \le n$.

Hence $(\nabla_X Y)(p) = \sum_{k=1}^{n} X(y_k)(p)P_k(t_0) = \frac{d}{dt}\left(P^{-1}_{c,t_0,t}(Y(c(t)))\right)\Big|_{t=t_0}$. Q.E.D.

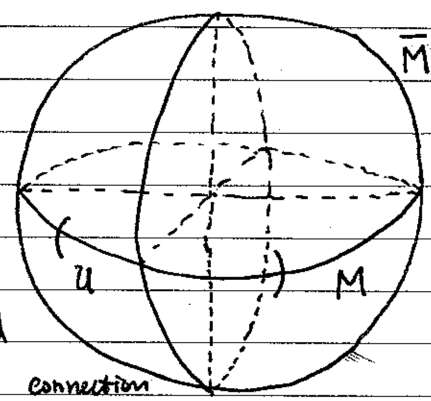
3. Let $f: M^n \longrightarrow \overline{M}^{n+k}$ be an immersion of a differentiable manifold M into a Riemannian manifold $\overline{M}$. Assume that M has the Riemannian metric induced by f (cf. Example 2.5 of Chap. 1). Let $p \in M$ and let $\mathcal{U} \subset M$ be a neighborhood of p such that $f(\mathcal{U}) \subset \overline{M}$ is a submanifold of $\overline{M}$. Further, suppose that X, Y are differentiable vector fields on $f(\mathcal{U})$ which extend to differentiable vector fields $\overline{X}, \overline{Y}$ on an open set of $\overline{M}$. Define

$$(\nabla_X Y)(p) = \text{tangential component of } (\overline{\nabla}_{\overline{X}} \overline{Y})(p),$$

where $\overline{\nabla}$ is the Riemannian connection of $\overline{M}$. Prove that ∇ is the Riemannian connection of M.

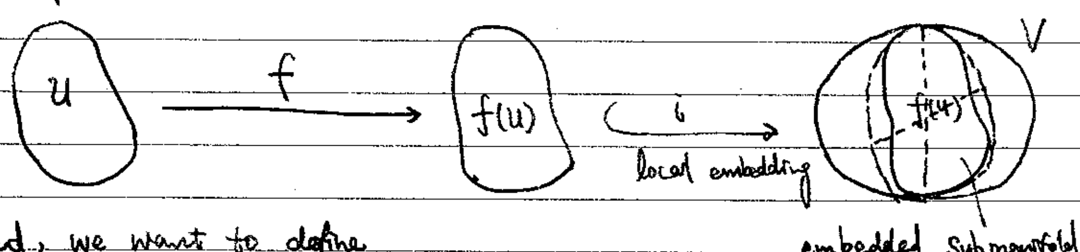
- Proof: $\boxed{0^\circ}$ A picture in mind:

$f(\mathcal{U})$ is not necessarily open in $\overline{M}$.



$\boxed{1^\circ}$ What is the problem saying?

∇ as stated in the problem is defined on $f(\mathcal{U})$. But we want a Riemannian connection ∇^M defined on M.



Indeed, we want to define

$$\langle \nabla^M_X Y, Z \rangle_{p,M} := \left\langle (f_*)^{-1} (\nabla_{f_*X} f_*Y)(p), Z \right\rangle_{p,M} = \left\langle (f_*)^{-1} \left(\overline{\nabla}_{\overline{f_*X}} \overline{f_*Y}\right)^{\text{tangent}}(p), Z \right\rangle_{p,M}$$

$$\overset{\text{by def. of induced metric}}{=} \left\langle \left(\overline{\nabla}_{\overline{f_*X}} \overline{f_*Y}\right)^{\text{tangent}}(p), \overline{f_*Z} \right\rangle_{f(p),\overline{M}} = \left\langle \overline{\nabla}_{\overline{f_*X}} \overline{f_*Y}, \overline{f_*Z} \right\rangle_{f(p),\overline{M}}$$

for any $X, Y, Z \in \mathrm{Vect}(\mathcal{U})$

$f_*X, f_*Y, f_*Z \in \mathrm{Vect}(f(\mathcal{U}))$. $f|_{\mathcal{U}}: \mathcal{U} \to f(\mathcal{U})$ is an embedding.

$\overline{f_*X}, \overline{f_*Y}, \overline{f_*Z} \in \mathrm{Vect}(V)$.

$\boxed{2^\circ}$ We want to clarify what an extension of a vector field on an embedded submanifold is, and how to describe it.

Let $X \in \mathrm{Vect}(\mathcal{U})$, and denote $\varphi_t(p)$ for its local flow:

$$\begin{cases} \dfrac{d}{dt}\varphi_t(p) = X(\varphi_t(p)) & \forall\, p \in \mathcal{U} \\ \varphi_0(p) = p \end{cases}$$

Pushing everything forward to $f(\mathcal{U})$:

$$\begin{cases} \dfrac{d}{dt}(f\circ\varphi_t\circ f^{-1})(f(p)) = \dfrac{d}{dt}(f\circ\varphi_t)(p) = f_*\dfrac{d}{dt}\varphi_t(p) = (f_*X)\big((f\circ\varphi_t\circ f^{-1})(f(p))\big) \\ (f\circ\varphi_0\circ f^{-1})(f(p)) = f(p) \qquad\qquad \forall\, p\in\mathcal{U} \end{cases}$$

Let $\psi_t(q) : V \longrightarrow V$ be a smooth funtion which extends $(f\circ\varphi_t\circ f^{-1})(f(p)) : f(\mathcal{U}) \to f(\mathcal{U})$ i.e. $\psi_t \circ f|_{\mathcal{U}} = f\circ\varphi_t|_{\mathcal{U}}$. $\psi_t(q)$ exists because we can write $f(\mathcal{U})$ in terms of canonical embedded coordinates $(x_1, \cdots, x_k, 0, \cdots, 0)$ while $V \supset f(\mathcal{U})$ has coordinates $(x_1, \cdots, x_k, x_{k+1}, \cdots, x_n)$. Define $\psi_t(q)$ to be $(f\circ\varphi_t\circ f^{-1})(f(p))$ on $f(\mathcal{U})$ and make $\psi_t(q)$ independent of the coordinates $(x_{k+1}, \cdots, x_n)$ on V.

Then we can define $\bar{X} \in \mathrm{Vect}(V)$ as

$$\begin{cases} \dfrac{d}{dt}\psi_t(q) = \bar{X}(\psi_t(q)) & \forall\, q\in V \\ \psi_0(q) = q \end{cases}$$

To see this is well-defined as an extension of $f_*X \in \mathrm{Vect}(f(\mathcal{U}))$, note that $\psi_t|_{f(\mathcal{U})} = f\circ\varphi_t\circ f^{-1}|_{f(\mathcal{U})}$, and hence

$$(f_*X)\big((f\circ\varphi_t\circ f^{-1})(f(p))\big) = \frac{d}{dt}(f\circ\varphi_t\circ f^{-1})(f(p)) = \frac{d}{dt}\psi_t(f(p)) = \bar{X}(\psi_t(f(p)))$$

(This section can be compared with J. Lee "Riemann Manifolds" P15-16, Exercise 2.3)

$$= \bar{X}(\psi_t(f(p))) = \bar{X}\big((f\circ\varphi_t\circ f^{-1})(f(p))\big).$$

$\boxed{3^\circ}$ Now we want to show that some quantities are independent of the choices of extensions.

First, let $X, Y \in \mathrm{Vect}(\mathcal{U})$. Then $f_*X, f_*Y \in \mathrm{Vect}(f(\mathcal{U}))$. We want to show that $[\overline{f_*X}, \overline{f_*Y}](f(p)) = [f_*X, f_*Y](f(p))$. By P.28 Proposition 5.4,

$[f_*X, f_*Y](f(p)) = \lim\limits_{t\to0}\frac{1}{t}\big[f_*Y - d(f\circ\varphi_t\circ f^{-1})(f_*Y)\big]\big((f\circ\varphi_t\circ f^{-1})(\psi_t(f(p)))\big)$ and $[\overline{f_*X}, \overline{f_*Y}](f(p))$

$$= \lim_{t\to0}\frac{1}{t}\big[\overline{f_*Y} - d\psi_t(\overline{f_*Y})\big](\psi_t(f(p))).$$

Note that $\overline{f_*Y}(\psi_t(f(p))) = (f_*Y)\left((f\circ\varphi_t\circ f^{-1})(f(p))\right)$ by definition of extension, and that for any $g\in C^\infty(V,\mathbb{R})$ there holds

$$\left(d\psi_t(\overline{f_*Y})\,g\right)(\psi_t(f(p))) = \overline{f_*Y}(g\circ\psi_t)(f(p)) = \overline{f_*Y}\,(g\circ f\circ\varphi_t\circ f^{-1})\,(f(p))$$

$$\underset{\text{because } \overline{f_*Y}(f(p)) = f_*Y(f(p))}{=} f_*Y\,(g\circ f\circ\varphi_t\circ f^{-1})(f(p)) = \left(d(f\circ\varphi_t\circ f^{-1})(f_*Y)\,g\right)(f\circ\varphi_t\circ f^{-1}(f(p))),$$

which gives $d\psi_t(\overline{f_*Y})\,(\psi_t(f(p))) = d(f\circ\varphi_t\circ f^{-1})(f_*Y)\left((f\circ\varphi_t\circ f^{-1})(\psi_t(f(p)))\right)$.

Thus $\frac{1}{t}\left[f_*Y - d(f\circ\varphi_t\circ f^{-1})(f_*Y)\right]\left((f\circ\varphi_t\circ f^{-1})(\psi_t(f(p)))\right) = \frac{1}{t}\left[\overline{f_*Y} - d\psi_t(\overline{f_*Y})\right](\psi_t(f(p)))$.

Taking limits with respect to $t\to 0$ on both sides gives

$$[\overline{f_*X},\overline{f_*Y}](f(p)) = [f_*X, f_*Y](f(p))$$

and hence

$$(*)\quad \left\langle [\overline{f_*X},\overline{f_*Y}],\ \overline{f_*Z}\right\rangle_{f(p),\overline{M}} = \left\langle [f_*X, f_*Y],\ f_*Z\right\rangle_{f(p),\overline{M}} \quad \text{for any } X,Y,Z\in Vect(\mathcal{U}).$$

Second, since $\overline{f_*X}(f(p)) = f_*X(f(p))$ and $\langle\overline{f_*Y},\overline{f_*Z}\rangle_{\overline{M}}(f(p)) = \langle f_*Y, f_*Z\rangle_{\overline{M}}(f(p))$ (i.e. $\langle\overline{f_*Y},\overline{f_*Z}\rangle_{\overline{M}}\big|_{f(\mathcal{U})} = \langle f_*Y, f_*Z\rangle_{\overline{M}}\big|_{f(\mathcal{U})}$), and that we can construct a local flow of $f_*X(f(p)) = \overline{f_*X}(f(p))$ which lies completely within $f(\mathcal{U})$, we have (by the definition of vector fields as a directional derivative) that

$$\overline{f_*X}(f(p))\langle\overline{f_*Y},\overline{f_*Z}\rangle_{f(p),\overline{M}} = \overline{f_*X}(f(p))\langle f_*Y, f_*Z\rangle_{f(p),\overline{M}} = f_*X(f(p))\langle f_*Y, f_*Z\rangle_{f(p),\overline{M}}.$$

or concisely

$$(**)\quad \overline{f_*X}\langle\overline{f_*Y},\overline{f_*Z}\rangle_{\overline{M}}\Big|_{f(\mathcal{U})} = f_*X\langle f_*Y, f_*Z\rangle_{\overline{M}}\Big|_{f(\mathcal{U})}.$$

Remark: We know $[X,Y] = \mathcal{L}_X Y$ is only a local operator on Y (not a tensor).
$: Y \longmapsto \mathcal{L}_X Y$

But actually it is better than a local operator: In order that $\mathcal{L}_X Y_1 = \mathcal{L}_X Y_2$ at p, we don't need $Y_1|_{\mathcal{U}} = Y_2|_{\mathcal{U}}$ on some neighborhood $\mathcal{U}$ of p. It suffices if Y_1 equals Y_2 along the local flow of X through p. In other words, only a small portion (measure zero) of points in a neighborhood of the point matters. This is essentially we established in section 3°.

$\boxed{4°}$. Since $\overline{\nabla}$ is a Levi-Civita connection on $\overline{M}$, we know from the proof of Theorem 3.6 that, for any $X, Y, Z \in Vect(\mathcal{U})$

$$
\begin{aligned}
\langle \nabla^M_X Y, Z\rangle_{p,M} &= \langle \overline{\nabla}_{\overline{f_*X}} \overline{f_*Y}, \overline{f_*Z}\rangle_{f(p),\overline{M}} \\
&= \frac{1}{2}\Big(\overline{f_*X}\langle \overline{f_*Y}, \overline{f_*Z}\rangle_{f(p),\overline{M}} + \overline{f_*Y}\langle \overline{f_*Z}, \overline{f_*X}\rangle_{f(p),M} - \overline{f_*Z}\langle \overline{f_*X}, \overline{f_*Y}\rangle_{f(p),M} \\
&\qquad - \langle [\overline{f_*X}, \overline{f_*Z}], \overline{f_*Y}\rangle_{f(p),\overline{M}} - \langle [\overline{f_*Y}, \overline{f_*Z}], \overline{f_*X}\rangle_{f(p),M} + \langle [\overline{f_*X}, \overline{f_*Y}], \overline{f_*Z}\rangle_{f(p),M}\Big) \\
&= \frac{1}{2}\Big(f_*X\langle f_*Y, f_*Z\rangle_{f(p),\overline{M}} + f_*Y\langle f_*Z, f_*X\rangle_{f(p),M} - f_*Z\langle f_*X, f_*Y\rangle_{f(p),M} \\
&\qquad - \langle [f_*X, f_*Z], f_*Y\rangle_{f(p),\overline{M}} - \langle [f_*Y, f_*Z], f_*X\rangle_{f(p),\overline{M}} + \langle [f_*X, f_*Y], f_*Z\rangle_{f(p),\overline{M}}\Big) \\
&= \frac{1}{2}\Big(X\langle Y, Z\rangle_{p,M} + Y\langle Z, X\rangle_{p,M} - Z\langle X, Y\rangle_{p,M} \\
&\qquad - \langle f_*[X,Z], f_*Y\rangle_{f(p),\overline{M}} - \langle f_*[Y,Z], f_*X\rangle_{f(p),\overline{M}} \\
&\qquad + \langle f_*[X,Y], f_*Z\rangle_{f(p),\overline{M}}\Big) \\
&= \frac{1}{2}\Big(X\langle Y, Z\rangle_{p,M} + Y\langle Z, X\rangle_{p,M} - Z\langle X, Y\rangle_{p,M} - \langle [X,Z], Y\rangle_{p,M} \\
&\qquad - \langle [Y,Z], X\rangle_{p,M} + \langle [X,Y], Z\rangle_{p,M}\Big)
\end{aligned}
$$

Again from the proof of Theorem 3.6, we know that the last formula uniquely determines the Levi-Civita connection on M. Here we used the observation

$$
X\langle Y, Z\rangle_{p,M} = X\big(\langle f_*Y, f_*Z\rangle_{f(p),\overline{M}} \circ f\big)(p) = f_*X\langle f_*Y, f_*Z\rangle_{f(p),\overline{M}}
$$

or if we put everything in coordinates,

$$
f_*X\langle f_*Y, f_*Z\rangle(f(p)) = \sum_{i=1}^n a^i(p)\frac{\partial y^l}{\partial x_i}\Big|_p \frac{\partial}{\partial y_l}\Big|_{f(p)} \sum_{j,k=1}^n b_j(p)\, c_k(p)\, g_{jk}(p) = \sum_{i=1}^n a^i(p)\frac{\partial}{\partial x_i}\Big|_p\Big(\sum_{j,k=1}^n b_j(p)c_k(p)g_{jk}(p)\Big) = X\langle Y, Z\rangle_{p,M}
$$

where we used $f_*Y(f(p)) = b_j(p)\frac{\partial y_m}{\partial x_j}\Big|_p \frac{\partial}{\partial y_m}\Big|_{f(p)}$, $f_*Z(f(p)) = c_k(p)\frac{\partial y_s}{\partial x_i}\Big|_p \frac{\partial}{\partial y_s}\Big|_{f(p)}$, and that

$$
g_{ij}(p) = \Big\langle \frac{\partial}{\partial x_i}\Big|_p, \frac{\partial}{\partial x_j}\Big|_p\Big\rangle_{p,M} = \Big\langle f_*\Big(\frac{\partial}{\partial x_i}\Big|_p\Big), f_*\Big(\frac{\partial}{\partial x_j}\Big|_p\Big)\Big\rangle_{f(p),\overline{M}} = \Big\langle \frac{\partial y_l}{\partial x_i}\Big|_p \frac{\partial}{\partial y_l}\Big|_{f(p)}, \frac{\partial y_k}{\partial x_j}\Big|_p \frac{\partial}{\partial y_k}\Big|_{f(p)}\Big\rangle_{f(p),\overline{M}}
$$

(Also that $\langle f_*Y, f_*Z\rangle_{f(p),\overline{M}} = \langle Y, Z\rangle_{p,M}$ by definition.)

$$
= \sum_{l,k=1}^n \frac{\partial y_l}{\partial x_i}\Big|_p \frac{\partial y_k}{\partial x_j}\Big|_p \overline{g}_{lk}(f(p)). \quad \blacksquare
$$

$\boxed{5°}$ One thing we forgot to mention in 4° is that we need to verify that ∇^M is indeed a connection on M. This is not difficult but an indispensible part in the proof, since we need ∇^M to first be a connection on M so that we can apply Theorem 3.6. Here we verify this according to (i)–(iii) in Definition 2.1.

(i) By definition, for any $X_1, X_2 \in \mathfrak{X}(M)$, $\overline{f_*X_1} + \overline{f_*X_2}$ is an extension of $f_*X_1 + f_*X_2$. Thus $\overline{\nabla}_{\overline{f_*X_1+f_*X_2}}\overline{f_*Y} = \overline{\nabla}_{\overline{f_*X_1}+\overline{f_*X_2}}\overline{f_*Y} = \overline{\nabla}_{\overline{f_*X_1}}\overline{f_*Y} + \overline{\nabla}_{\overline{f_*X_2}}\overline{f_*Y}$ and for all $X, Y, Z \in \mathfrak{X}(M)$

$$\left\langle \nabla^M_{X_1+X_2}Y, Z\right\rangle_{p,M} = \left\langle (f_*)^{-1}\left(\nabla_{f_*(X_1+X_2)}f_*Y\right)(p), Z\right\rangle_{p,M} = \left\langle (f_*)^{-1}\left(\overline{\nabla}_{\overline{f_*(X_1+X_2)}}\overline{f_*Y}\right)^{\text{tangent}}(p), Z\right\rangle_{p,M}$$

$$= \left\langle \left(\overline{\nabla}_{\overline{f_*X_1+f_*X_2}}\overline{f_*Y}\right)^{\text{tangent}}(p), f_*Z(p)\right\rangle_{f(p),\overline{M}} = \left\langle \left(\overline{\nabla}_{\overline{f_*X_1}}\overline{f_*Y} + \overline{\nabla}_{\overline{f_*X_2}}\overline{f_*Y}\right)^{\text{tangent}}(p), f_*Z(p)\right\rangle_{f(p),\overline{M}}$$

$$= \left\langle \left(\overline{\nabla}_{\overline{f_*X_1}}\overline{f_*Y}\right)^{\text{tangent}}(p) + \left(\overline{\nabla}_{\overline{f_*X_2}}\overline{f_*Y}\right)^{\text{tangent}}(p), f_*Z(p)\right\rangle_{f(p),\overline{M}}$$

$$= \left\langle \left(\overline{\nabla}_{\overline{f_*X_1}}\overline{f_*Y}\right)^{\text{tangent}}(p), \overline{f_*Z}(p)\right\rangle_{f(p),\overline{M}} + \left\langle \left(\overline{\nabla}_{\overline{f_*X_2}}\overline{f_*Y}\right)^{\text{tangent}}(p), \overline{f_*Z}(p)\right\rangle_{f(p),M}$$

$$= \left\langle (f_*)^{-1}\left(\nabla_{f_*X_1}f_*Y\right)(p), Z\right\rangle_{p,M} + \left\langle (f_*)^{-1}\left(\nabla_{f_*X_2}f_*Y\right)(p), Z\right\rangle_{p,M}$$

$$= \left\langle \nabla^M_{X_1}Y, Z\right\rangle_{p,M} + \left\langle \nabla^M_{X_2}Y, Z\right\rangle_{p,M} \quad \left(= \left\langle \nabla^M_{X_1}Y + \nabla^M_{X_2}Y, Z\right\rangle_{p,M}\right)$$

Similarly, by definition, for any $g \in C^\infty(M)$, $\overline{f_*g}\cdot\overline{f_*X}$ is an extension of $f_*g\cdot f_*X$. Hence by a similar argument as above one can show that

$$\left\langle \nabla^M_{gX}Y, Z\right\rangle_{p,M} = \left\langle g\nabla^M_X Y, Z\right\rangle_{p,M}.$$

Therefore $\nabla^M_X Y$ is $C^\infty(M)$-linear in X. (Here we used $\left(\overline{f_*g}\cdot\overline{\nabla}_{\overline{f_*X}}\overline{f_*Y}\right)(f(p)) = \left(f_*g\cdot\nabla_{f_*X}f_*Y\right)(f(p)) = f_*\left((g\nabla^M_X Y)(p)\right)$)

(ii) That $\nabla^M_X Y$ is additive in Y follows from a similar argument as in (i).

(iii) For any $g \in C^\infty(M)$, note that by definition $\overline{f_*g}\cdot\overline{f_*Y}$ is an extension of $f_*g\cdot f_*Y$. Thus $\overline{\nabla}_{\overline{f_*X}}\overline{f_*g\cdot f_*Y} = \overline{\nabla}_{\overline{f_*X}}\overline{f_*g}\cdot\overline{f_*Y} = \overline{f_*g}\,\overline{\nabla}_{\overline{f_*X}}\overline{f_*Y} + \left[\overline{f_*X}\left(\overline{f_*g}\right)\right]\overline{f_*Y}$. It then follows from a similar argument as above that

$$\left\langle \nabla^M_X(gY), Z\right\rangle_{p,M} = \left\langle g\nabla^M_X Y + X(g)Y, Z\right\rangle_{p,M},$$

where we used $\overline{f_*X}\left(\overline{f_*g}\right) = \overline{f_*\left(X(g)\right)}$, i.e. $\overline{f_*X}\left(\overline{f_*g}\right)$ is an extension of $f_*(X(g))$.

Remark: In solving this problem, we could have locally identified M with its image in $\overline{M}$, and thus ∇^M with ∇, to simplify our notation; it was the Obssesive-Compulsive Disorder that prevented us from doing so (or equivalently speaking, "being too pedantic"). In Chapter 6

No.

Date

this "extension of an push-forward" type construction is frequently used, and the simplification of notation mentioned above is heavily used or implicitly assumed. Of course we were unaware of it when we were writing the solution 0°–4° to this problem on January 15th, 2012. This forced 5° to be kept to the notations set up in 1°, although 5° is written after reading part of Chapter 6.

4. Let $M^2 \subset \mathbb{R}^3$ be a surface in $\mathbb{R}^3$ with the induced Riemannian metric. Let $c: I \longrightarrow M$ be a differentiable curve on M and let V be a vector field tangent to M along c; V can be thought of as a smooth function $V: I \longrightarrow \mathbb{R}^3$, with $V(t) \in T_{c(t)}M$.

a) Show that V is parallel if and only if $\frac{dV}{dt}$ is perpendicular to $T_{c(t)}M \subset \mathbb{R}^3$ where $\frac{dV}{dt}$ is the usual derivative of $V: I \longrightarrow \mathbb{R}^3$!

b) If $S^2 \subset \mathbb{R}^3$ is the unit sphere of $\mathbb{R}^3$, show that the velocity field along great circles, parametrized by arc length, is a parallel field. A similar argument holds for $S^n \subset \mathbb{R}^{n+1}$.

• Proof: (a) We shall make use of our machinery built in Problem 3.
Here the inclusion map $i: M^2 \hookrightarrow \mathbb{R}^3$ is a local embedding.
For any vector $Z \in T_{c(t)}M$, we can first extend it along the curve c by parallel transport, and then extend it to all of $\mathbb{R}^3$. Hence if $\frac{DV}{dt} = 0$, then $0 = \left\langle \frac{DV}{dt}, Z \right\rangle_{c(t), M} = \left\langle \nabla_{c'(t)} V, Z \right\rangle_{c(t), M} = \left\langle \overline{\nabla}_{\overline{i_* c'(t)}} \overline{i_* V}, \overline{i_* Z} \right\rangle_{c(t), \mathbb{R}^3}$

where $\overline{\nabla}$ is the standard Riemannian connection in $\mathbb{R}^3$, whose $\Gamma_{ij}^k \equiv 0 \ \forall i, j, k$.
It follows that $\overline{\nabla}_{\overline{i_* c'(t)}} \overline{i_* V} = \sum_{k=1}^{3} \frac{d v^k(c(t))}{dt} \frac{\partial}{\partial x_k}$ and hence

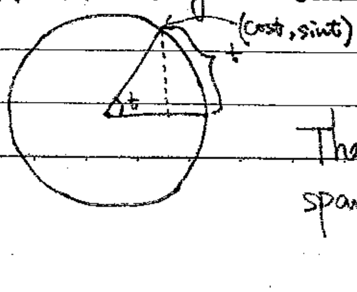
$$0 = \left\langle \frac{DV}{dt}, Z \right\rangle_{c(t), M} = \left\langle \overline{\nabla}_{\overline{i_* c'(t)}} \overline{i_* V}, \overline{i_* Z} \right\rangle_{c(t), \mathbb{R}^3} = \left\langle \sum_{k=1}^{3} \frac{d v^k(c(t))}{dt} \frac{\partial}{\partial x_k}, \sum_{j=1}^{3} z^j(c(t)) \frac{\partial}{\partial x_j} \right\rangle$$

$$= \sum_{j,k=1}^{3} \frac{d v^k(c(t))}{dt} z^j(c(t)) \delta_{jk} = \sum_{k=1}^{3} \frac{d v^k(c(t))}{dt} z^j(c(t))$$

$$= \left\langle \frac{dV}{dt}, Z \right\rangle_{c(t), \mathbb{R}^3}$$

which proves $\frac{dV}{dt} \perp T_{c(t)}M$. Reversing all previous steps gives us the "if" part.

(b) Choose good coordinates to parametrize the great circle by $(\cos t, \sin t, 0) \in \mathbb{R}^3$.

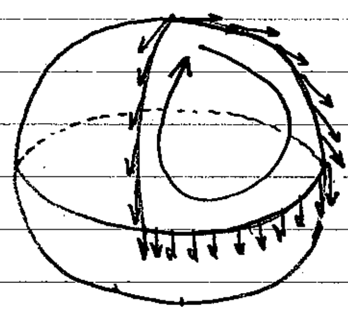


Then $V(t)$ = the velocity field of the great circle
$= (-\sin t, \cos t, 0) \in \mathbb{R}^3$
Then $\frac{dV}{dt} = (-\cos t, -\sin t, 0) \perp T_{c(t)}S^2$ because $T_{c(t)}S^2$ is spanned by $V(t) = (-\sin t, \cos t, 0)$ and $N(t) = (0, 0, 1)$.

A similar argument holds for $S^n \subset \mathbb{R}^{n+1}$. To see this, first note that the great circle on S^n lies in a two-dimensional subspace of $\mathbb{R}^{n+1}$ by the symmetry of S^n (cf. Ng. Math 267 Final Exam Problem 2 b)). Then we can parametrise the great circle by $(\sin t, \cos t, 0, \cdots, 0)$ once we pick sufficiently good coordinates. Then all that is remained is almost identical to the case $S^2 \subset \mathbb{R}^3$. ∎

5. In Euclidean space, the parallel transport of a vector between two points does not depend on the curve joining the two points. Show, by example that this fact may not be true on an arbitrary Riemannian manifold.

- Solution:



It is already shown in Problem 4 b) that $V(t) = (-\sin t, \cos t, 0)$, which is perpendicular to the position vector $(\cos t, \sin t, 0)$ and lies in the plane span$\{e_1, e_2\}$ with constant modulus, is a parallel vector field.

The same can be shown for $W(t) = (0, 0, 1)$. Thus the picture above makes sense and gives an example as desired.

6. Let M be a Riemannian manifold and let p be a point of M. Consider a constant curve $f: I \to M$ given by $f(t) = p$, for all $t \in I$. Let V be a vector field along f (that is, V is a differentiable mapping of I into T_pM). Show that $\frac{DV}{dt} = \frac{dV}{dt}$, that is to say, the covariant derivative coincides with the usual derivative of $V: I \to T_pM$.

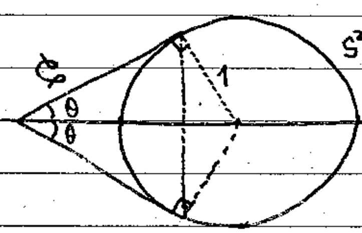
- Proof: Put everything into coordinates:

$$\frac{DV}{dt} = \sum_k \left(\frac{dv^k}{dt} + \sum_{i,j} \Gamma^k_{ij} v^j \frac{df^i}{dt} \right) \frac{\partial}{\partial x_k} = \sum_k \frac{dv^k}{dt} \frac{\partial}{\partial x^k} = \frac{d}{dt}\left(\sum_k v^k \frac{\partial}{\partial x^k} \right) = \frac{dV}{dt}.$$

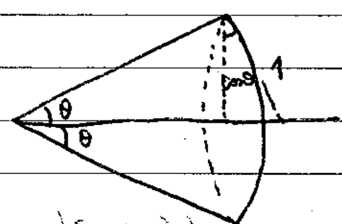
cf. do Carmo "Differential Geometry of Curves and Surfaces", P243 Example 1

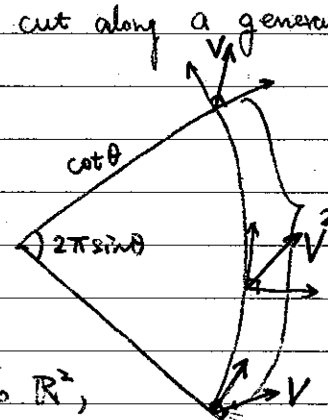
7. Let $S^2 \subset \mathbb{R}^3$ be the unit sphere, c an arbitrary parallel of latitude on S^2 and V_0 a tangent vector to S^2 at a point of c. Describe geometrically the parallel transport of V_0 along c.
Hint: Consider the cone $\mathcal{C}$ tangent to S^2 along c and show that the parallel transport of V_0 along c is the same, whether taken relative to S^2 or to $\mathcal{C}$.

- Solution: Along the latitude c, at each point $p \in C$ it is easy to verify that $T_p\mathcal{C} = T_p S^2$. (make use of the identical basis for both tangent planes.)



By the characterization of parallel vector fields given in Problem 4 a) and $T_{c(t)}\mathcal{C} = T_{c(t)}S^2$ we know the parallel transport of V_0 along c is the same whether taken relative to S^2 or to $\mathcal{C}$. Hence it suffices to consider the parallel transport on $\mathcal{C}$.
Since $\frac{dV}{dt}$ is perpendicular to $T_{c(t)}\mathcal{C}$, $V(t)$ does not change as a vector in the tangent plane. Hence it suffices to see how does a frame along C changes.
Note that the cone $\mathcal{C}$ when cut along a generating line is isometric to a sector area in $\mathbb{R}^2$.





When $\mathcal{C}$ {a ray} is mapped isometrically to $\mathbb{R}^2$,
it is then immediate to see that the frame for the tangent spaces along c simply rotates. Since V stays the same in each tangent plane, V is relatively static with respect to the frame (isometry does not change angles!). Thus the change of V along C, when considered as a sector region in $\mathbb{R}^2$, basically is a rotation at the same rate as that of the frame.

8. Consider the upper half-plane $\mathbb{R}^2_+ = \{(x,y) \in \mathbb{R}^2 ; y>0\}$ with the metric given by $g_{11} = g_{22} = \frac{1}{y^2}$, $g_{12} = 0$ (metric of Lobatchevski's non-euclidean geometry).

a) Show that the Christoffel symbols of the Riemannian connection are:
$\Gamma^1_{11} = \Gamma^2_{12} = \Gamma^1_{22} = 0$, $\Gamma^2_{11} = \frac{1}{y}$, $\Gamma^1_{12} = \Gamma^2_{22} = -\frac{1}{y}$.

b) Let $v_0 = (0,1)$ be a tangent vector at point $(0,1)$ of $\mathbb{R}^{2+}$ (v_0 is a unit vector on the y-axis with origin at $(0,1)$). Let $v(t)$ be the parallel transport of v_0 along the curve $x=t$, $y=1$. Show that $v(t)$ makes an angle t with the direction of the y-axis, measured in the clockwise sense.

Hint: The field $v(t) = (a(t), b(t))$ satisfies the system (2) (cf. P53) which defines a parallel field and which, in this sense, simplifies to

$$\begin{cases} \frac{da}{dt} + \Gamma^1_{12} b = 0, \\ \frac{db}{dt} + \Gamma^2_{11} a = 0. \end{cases}$$

Taking $a = \cos\theta(t)$, $b = \sin\theta(t)$ and noting that along the given curve we have $y=1$, we obtain from the equations above that $\frac{d\theta}{dt} = -1$. Since $v(0) = v_0$, this implies that $\theta(t) = \pi/2 - t$.

- Proof: a) It is easy to see that $g_{11} = g_{22} = 1/y^2$, $g_{12} = g_{21} = 0$; $g^{11} = g^{22} = y^2$, $g^{12} = g^{21} = 0$. It follows from direct computations that

$$\Gamma^1_{11} = \frac{1}{2}(\partial_1 g_{1l} + \partial_1 g_{l1} - \partial_l g_{11}) g^{l1} = -\frac{1}{2}\frac{\partial g_{11}}{\partial y} g^{21} = 0$$
$$\Gamma^2_{12} = \Gamma^2_{21} = \frac{1}{2}(\partial_1 g_{2l} + \partial_2 g_{l1} - \partial_l g_{12}) g^{l2} = \frac{1}{2}\frac{\partial g_{21}}{\partial y} g^{22} = 0$$
$$\Gamma^1_{22} = \frac{1}{2}(\partial_2 g_{2l} + \partial_2 g_{l2} - \partial_l g_{22}) g^{l1} = -\frac{1}{2}\frac{\partial g_{22}}{\partial x} g^{21} = 0$$
$$\Gamma^1_{12} = \Gamma^1_{21} = \frac{1}{2}(\partial_1 g_{2l} + \partial_2 g_{l1} - \partial_l g_{12}) g^{l1} = \frac{1}{2}\frac{\partial g_{11}}{\partial y} g^{11} = \frac{1}{2}\cdot(-2)\frac{1}{y^3}\cdot y^2 = -\frac{1}{y}$$
$$\Gamma^2_{11} = \frac{1}{2}(\partial_1 g_{1l} + \partial_1 g_{l1} - \partial_l g_{11}) g^{l2} = -\frac{1}{2}\frac{\partial g_{11}}{\partial y} g^{22} = -\frac{1}{2}\cdot(-2)\frac{1}{y^3}\cdot y^2 = \frac{1}{y}$$
$$\Gamma^2_{22} = \frac{1}{2}(\partial_2 g_{2l} + \partial_2 g_{l2} - \partial_l g_{22}) g^{l2} = \frac{1}{2}\frac{\partial g_{22}}{\partial y} g^{22} = \frac{1}{2}\cdot(-2)\frac{1}{y^3}\cdot y^2 = -\frac{1}{y}.$$

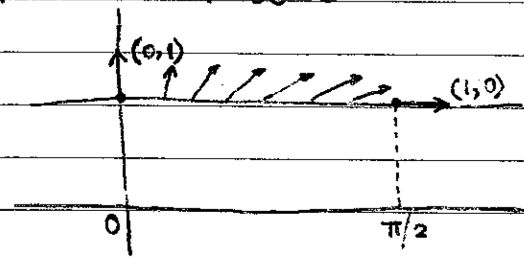
b) In coordinates, the parallel transport equations are (following the hint):

$$\begin{cases} 0 = \frac{dv^1}{dt} + \sum_{i,j}\Gamma^1_{ij} v^j \frac{dx_i}{dt} = \frac{da}{dt} + \sum_j \Gamma^1_{ij} v^j = \frac{da}{dt} - \frac{1}{y} b = \frac{da}{dt} - b \\ 0 = \frac{dv^2}{dt} + \sum_{i,j}\Gamma^2_{ij} v^j \frac{dx_i}{dt} = \frac{db}{dt} + \sum_j \Gamma^2_{ij} v^j = \frac{db}{dt} + \frac{1}{y} a = \frac{db}{dt} + a \\ a(0) = 0, \; b(0) = 1 \end{cases}$$

Solving this system of ODEs gives us $a(t) = \sin t$, $b(t) = \cos t$.
Hence the desired angle is given by

$$\theta(t) = \arccos\left(\frac{\langle (a(t), b(t)), (0,1)\rangle}{\|(a(t), b(t))\| \cdot \|(0,1)\|}\right) = \arccos(\cos t) = t$$

which is what to be shown.



9. (Pseudo-Riemannian Metrics). A pseudo-Riemannian metric on a smooth manifold M is a choice, at every point $p \in M$, of a non-degenerate symmetric bilinear form $\langle , \rangle$ (not necessarily positive definite) on T_pM which varies differentiably with p. Except for the fact that $\langle , \rangle$ need not be positive definite, all of the definitions that have been presented up to now make sense for a pseudo-Riemannian metric. For example, an affine connection on M compatible with a pseudo-Riemannian metric on M satisfies (4) (cf. P54); if, in addition, (5) (cf. P54) holds, the affine connection is said to be symmetric.
a) Show that the theorem of Levi-Civita extends to pseudo-Riemannian metrics. The connection so obtained is called the pseudo-Riemannian connection.
b) Introduce a pseudo-Riemannian metric on $\mathbb{R}^{n+1}$ by using the quadratic form:

$$Q(x_0, \cdots, x_n) = -(x_0)^2 + (x_1)^2 + \cdots + (x_n)^2, \quad (x_0, \cdots, x_n) \in \mathbb{R}^{n+1}.$$

Show that the parallel transport corresponding to the Levi-Civita connection of this metric coincides with the usual transport of $\mathbb{R}^{n+1}$ (this pseudo-Riemannian metric is called the Lorentz metric; for $n=3$, it appears naturally in relativity).

• Proof: a) Just note that we did not need positive-definiteness in the proof of Theorem 3.6. We only need to elaborate the proof in do Carmo a little bit. [Barrett O'Neill, "Semi-Riemannian Geometry with Applications to Relativity", Chapter 10, Proposition 10 (P.60) cited below:]

No.

Date

Proposition 10. Let M be a semi-Riemannian manifold. If $V \in \mathfrak{X}(M)$, let V^* be the one-form on M such that $V^*(X) = \langle V, X\rangle$ for all $X \in \mathfrak{X}(M)$. Then the function $V \longrightarrow V^*$ is a $C^\infty(M,\mathbb{R})$-linear isomorphism from $\mathfrak{X}(M)$ to $\mathfrak{X}^*(M)$.

Proof of Proposition 10. Since V^* is $C^\infty(M)$-linear it is indeed a one-form, and the function $V \longrightarrow V^*$ is also $C^\infty(M)$-linear. That it is an isomorphism follows from two facts:

(a) If $\langle V, X\rangle = \langle W, X\rangle$ for all $X \in \mathfrak{X}(M)$, then $V = W$.

(b) Given any one-form $\theta \in \mathfrak{X}^*(M)$ there is a unique vector field $V \in \mathfrak{X}(M)$ such that $\theta(X) = \langle V, X\rangle$ for all X.

Let $U = V - W$. Then assertion (a) amounts to showing that if $\langle U_p, X_p\rangle = 0$ for all $X \in \mathfrak{X}(M)$ and all $p \in M$, then $U = 0$. Since every element of $T_p(M)$ has the form X_p, the result follows by the nondegeneracy of the metric tensor.

Now (a) is exactly the uniqueness assertion in (b), hence to prove (b) it suffices to find V on an arbitrary coordinate neighborhood $\mathcal{U}$. (All these local V's will be consistent on overlaps.) If $\theta = \sum \theta_i dx^i$ on $\mathcal{U}$, let $V = \sum_{i,j} g^{ij}\theta_i \partial_j$. Then since (g_{ij}) and (g^{ij}) are inverse matrices,

$$\langle V, \partial_k\rangle = \sum_{i,j} g^{ij}\theta_i \langle \partial_j, \partial_k\rangle = \sum_{i,j} \theta_i g^{ij} g_{jk} = \sum_i \theta_i \delta_{ik} = \theta_k = \theta(\partial_k).$$

It follows by $C^\infty(M)$-linearity that $\langle V, X\rangle = \theta(X)$ for all X on $\mathcal{U}$. ∎

Now we return to the proof of (a). The Koszul formula is used:

$$Z \longmapsto \frac{1}{2}\Big(X\langle Y, Z\rangle + Y\langle Z, X\rangle - Z\langle X, Y\rangle - \langle [X,Z], Y\rangle - \langle [Y,Z], X\rangle + \langle [X,Y], Z\rangle \Big)$$

By formulae $[fX, Y] = f[X,Y] - Y(f)X$ and $[X, fY] = f[X,Y] + X(f)Y$, it is direct to verify that the mapping above is a tensor. Then by the non-degeneracy we conclude that it uniquely defines a vector field on M, which is then denoted as $\nabla_X Y$. All the remaining properties are easy to verify. (cf. Ng. Homework 4, Problem 1.)

b) It suffices to show that $\Gamma_{ij}^k = 0 \quad \forall\, 1 \le i, j, k \le n+1$ for the Lorentz metric. This is obvious since as we did for the Riemannian metrics one can show that in coordinates

$$\Gamma_{ij}^m = \frac{1}{2} \sum_k \left(\frac{\partial}{\partial x_i} g_{jk} + \frac{\partial}{\partial x_j} g_{ki} - \frac{\partial}{\partial x_k} g_{ij} \right) g^{km}$$

and for the Lorentz metric we know $g_{ij} \equiv \text{const}_{ij}$ for $1 \le i, j \le n+1$. (see Barrett O'Neill, "Semi-Riemannian Geometry with Applications to Relativity", Chapter 10, Lemma 14 (P.62)) Then by seeing the parallel transport ODEs on P.53 we get what we want.

No.

Date

cf. do Carmo "Differential Geometry of Curves and Surfaces", P.255 Example 5.

1. (Geodesics of a surface of revolution). Denote by (u,v) the cartesian coordinates of $\mathbb{R}^2$. Show that the function $\varphi: U \subset \mathbb{R}^2 \to \mathbb{R}^3$ given by
$\varphi(u,v) = (f(v)\cos u,\ f(v)\sin u,\ g(v))$, $U = \{(u,v) \in \mathbb{R}^2 : u_0 < u < u_1;\ v_0 < v < v_1\}$,
where f and g are differentiable functions, with $f'(v)^2 + g'(v)^2 \neq 0$ and $f(v) \neq 0$, is an immersion. The image $\varphi(U)$ is the surface generated by the rotation of the curve $(f(v), g(v))$ around the axis Oz and is called a surface of revolution S. The image by φ of the curves $u =$ const and $v =$ const (parametrized by arc length) are called meridians and parallels, respectively, of S.

a) Show that the induced metric in the coordinates (u,v) is given by
$g_{11} = f^2,\quad g_{12} = 0,\quad g_{22} = (f')^2 + (g')^2.$

b) Show that local equations of a geodesic γ are

$$\frac{du^2}{dt^2} + \frac{2ff'}{f^2}\frac{du}{dt}\frac{dv}{dt} = 0,$$

$$\frac{dv^2}{dt^2} - \frac{ff'}{(f')^2+(g')^2}\left(\frac{du}{dt}\right)^2 + \frac{f'f'' + g'g''}{(f')^2+(g')^2}\left(\frac{dv}{dt}\right)^2 = 0$$

c) Obtain the following geometric meaning of the equations above: the second equation is, except for meridians and parallels, equivalent to the fact that the "energy" $|\gamma'(t)|^2$ of a geodesic is constant along γ; the first equation signifies that if $\beta(t)$ is the oriented angle, $\beta(t) < \pi$, of γ with a parallel P intersecting γ at $\gamma(t)$, then

$$r\cos\beta = \text{Const.},$$

where r is the radius of the parallel P (the equation above is called Clairaut's relation).

d) Use Clairaut's relation to show that a geodesic of the paraboloid
$(f(v) = v,\ g(v) = v^2,\ 0 < v < \infty,\ +\varepsilon < u < 2\pi + \varepsilon)$
which is not a meridian, intersects itself an infinite number of times (Fig. 6).

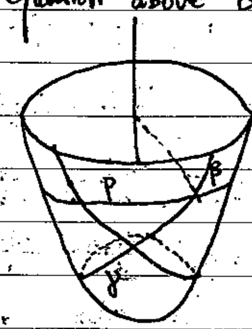


Figure 6. Geodesics of a paraboloid

$$\Gamma_{ij}^{m} = \frac{1}{2}\sum_{k} g^{km}\left(\partial_i g_{jk} + \partial_j g_{ki} - \partial_k g_{ij}\right)$$

No.

Date

- **Proof:** $\varphi(u,v) = \left(f(v)\cos u,\ f(v)\sin u,\ g(v)\right)$

$$\Rightarrow d\varphi(u,v) = \begin{pmatrix} -f(v)\sin u, & f(v)\cos u, & 0 \\ f'(v)\cos u, & f'(v)\sin u, & g'(v) \end{pmatrix}$$

The determinants of all 2×2 sub-matrices are

$$\det A_{12} = -f(v)f'(v)\sin^2 u - f(v)f'(v)\cos^2 u = -f(v)f'(v)$$

$$\det A_{13} = -f(v)g'(v)\sin u$$

$$\det A_{23} = f(v)g'(v)\cos u$$

So $\det A_{12}$, $\det A_{13}$, $\det A_{23}$ can not all vanish, since

$$|\det A_{12}|^2 + |\det A_{13}|^2 + |\det A_{23}|^2 = [f(v)f'(v)]^2 + [f(v)g'(v)]^2 = [f(v)]^2[f'(v)^2 + g'(v)^2]$$

and by assumption we have $f(v) \neq 0$, $f'(v)^2 + g'(v)^2 \neq 0$.

a) $\varphi_u = \left(-f(v)\sin u,\ f(v)\cos u,\ 0\right)$, $\varphi_v = \left(f'(v)\cos u,\ f'(v)\sin u,\ g'(v)\right)$

$$\Rightarrow g_{11} = \langle \varphi_u, \varphi_u\rangle_{\mathbb{R}^3} = f(v)^2\sin^2 u + f(v)^2\cos^2 u = f(v)^2$$

$$g_{21} = g_{12} = \langle \varphi_u, \varphi_v\rangle_{\mathbb{R}^3} = -f(v)f'(v)\sin u\cos u + f(v)f'(v)\sin u\cos u = 0$$

$$g_{22} = \langle \varphi_v, \varphi_v\rangle_{\mathbb{R}^3} = f'(v)^2\cos^2 u + f'(v)^2\sin^2 u + g'(v)^2 = f'(v)^2 + g'(v)^2$$

b) Recall that the geodesic equations are of the form

$$\frac{d^2u}{dt^2} + \Gamma^1_{11}\left(\frac{du}{dt}\right)^2 + \Gamma^1_{22}\left(\frac{dv}{dt}\right)^2 + 2\Gamma^1_{12}\frac{du}{dt}\frac{dv}{dt} = 0$$

$$\frac{d^2v}{dt^2} + \Gamma^2_{11}\left(\frac{du}{dt}\right)^2 + \Gamma^2_{22}\left(\frac{dv}{dt}\right)^2 + 2\Gamma^2_{12}\frac{du}{dt}\frac{dv}{dt} = 0$$

We deduce from (a) that (noting that $g^{11} = 1/f(v)^2$, $g^{12} = g^{21} = 0$, $g^{22} = 1/[f'(v)^2 + g'(v)^2]$)

$$\Gamma^1_{11} = \frac{1}{2}g^{11}\left(\frac{\partial g_{11}}{\partial u} + \frac{\partial g_{11}}{\partial u} - \frac{\partial g_{11}}{\partial u}\right) = 0$$

$$\Gamma^1_{12} = \frac{1}{2}g^{11}\left(\frac{\partial g_{21}}{\partial u} + \frac{\partial g_{11}}{\partial v} - \frac{\partial g_{12}}{\partial u}\right) = \frac{1}{2f(v)^2}\cdot 2f(v)f'(v) = \frac{f(v)f'(v)}{f(v)^2}$$

$$\Gamma^1_{22} = \frac{1}{2}g^{11}\left(\frac{\partial g_{21}}{\partial u} + \frac{\partial g_{12}}{\partial v} - \frac{\partial g_{22}}{\partial u}\right) = 0$$

$$\Gamma^2_{11} = \frac{1}{2}g^{22}\left(\frac{\partial g_{12}}{\partial u} + \frac{\partial g_{12}}{\partial u} - \frac{\partial g_{11}}{\partial v}\right) = -\frac{1}{2[f'(v)^2 + g'(v)^2]}\cdot 2f(v)f'(v) = \frac{f(v)f'(v)}{f'(v)^2 + g'(v)^2}$$

$$\Gamma^2_{12} = \frac{1}{2}g^{22}\left(\frac{\partial g_{22}}{\partial u} + \frac{\partial g_{21}}{\partial v} - \frac{\partial g_{12}}{\partial v}\right) = 0$$

$$\Gamma^2_{22} = \frac{1}{2}g^{22}\left(\frac{\partial g_{22}}{\partial v} + \frac{\partial g_{22}}{\partial v} - \frac{\partial g_{22}}{\partial v}\right) = \frac{1}{2[f'(v)^2 + g'(v)^2]}\left(2f'(v)f''(v) + 2g'(v)g''(v)\right)$$

$$= \frac{f'(v)f''(v) + g'(v)g''(v)}{f'(v)^2 + g'(v)^2}$$

Substituting these Christoffel symbols into the geodesic equations, we obtain

$$\frac{d^2u}{dt^2}+\frac{2ff'}{f^2}\frac{du}{dt}\frac{dv}{dt}=0 \qquad ①$$

$$\frac{d^2v}{dt^2}+\frac{ff'}{(f')^2+(g')^2}\left(\frac{du}{dt}\right)^2+\frac{f'f''+g'g''}{(f')^2+(g')^2}\left(\frac{dv}{dt}\right)^2=0 \qquad ②$$

c) Assume $\gamma(t)=\left(f(v(t))\cos u(t),\ f(v(t))\sin u(t),\ g(v(t))\right)$, then

$$\gamma'(t)=\left(f'(v)v'(t)\cos u(t)-f(v)\sin u(t)\,u'(t),\ f'(v)v'(t)\sin u(t)+f(v)\cos u(t)\,u'(t),\ g'(v)v'(t)\right)$$

$$\Rightarrow |\gamma'(t)|^2=(f'(v)v'(t))^2+(f(v)u'(t))^2+(g'(v)v'(t))^2=f(v)^2\left(\frac{du}{dt}\right)^2+\left(f'(v)^2+g'(v)^2\right)\left(\frac{dv}{dt}\right)^2$$

thus $$\frac{d|\gamma'(t)|^2}{dt}=2f\frac{du}{dt}\left(f'\frac{dv}{dt}\frac{du}{dt}+f\frac{d^2u}{dt^2}\right)+\left(2f'f''+2g'g''\right)\left(\frac{dv}{dt}\right)^3+\left((f')^2+(g')^2\right)\cdot 2\frac{dv}{dt}\frac{d^2v}{dt^2}$$

$$\overset{\text{by ①}}{=\!=}2f\frac{du}{dt}\left(f'\frac{dv}{dt}\frac{du}{dt}-2f'\frac{du}{dt}\frac{dv}{dt}\right)+2\left[(f')^2+(g')^2\right]\frac{dv}{dt}\left(\frac{f'f''+g'g''}{(f')^2+(g')^2}\left(\frac{dv}{dt}\right)^2+\frac{d^2v}{dt^2}\right)$$

$$=-2ff'\frac{dv}{dt}\left(\frac{du}{dt}\right)^2+2\left[(f')^2+(g')^2\right]\frac{dv}{dt}\left(\frac{f'f''+g'g''}{(f')^2+(g')^2}\left(\frac{dv}{dt}\right)^2+\frac{d^2v}{dt^2}\right)$$

$$=2\left[(f')^2+(g')^2\right]\frac{dv}{dt}\left(\frac{d^2v}{dt^2}-\frac{ff'}{(f')^2+(g')^2}\left(\frac{du}{dt}\right)^2+\frac{f'f''+g'g''}{(f')^2+(g')^2}\left(\frac{dv}{dt}\right)^2\right)$$

Hence ② $\Rightarrow \frac{d|\gamma'(t)|^2}{dt}=0 \Leftrightarrow |\gamma'(t)|$ is constant along γ. On the other hand, if γ is not a parallel, then $\frac{dv}{dt}\neq 0$ and hence $\frac{d|\gamma'(t)|^2}{dt}=0 \Rightarrow$ ② holds.

(I can't see why γ can not be a meridian. Indeed, as we are going to show below, all meridians which are parametrized by arc-length are geodesics, and hence $|\gamma'(t)|=\text{const}$. But situations are more restrictive for parallels: only some parallels are geodesics. See below, or cf. do Carmo "Differential Geometry of Curves and Surfaces" P255 Example 5.)

1°. We point out that the meridians $u=\text{const.}$, $v=v(s)$, parametrized by arc length s, are all geodesics. Note that ① holds trivially, and the fact that it is parametrized by arc length suggests the tangent vector is of unit length, i.e. $|\gamma'(t)|^2=(f'(v)v'(t))^2+(g'(v)v'(t))^2=(f'(v)^2+g'(v)^2)\left(\frac{dv}{dt}\right)^2$, which gives

$$\left(\frac{dv}{dt}\right)^2=\frac{1}{(f')^2+(g')^2}\Rightarrow 2\frac{dv}{dt}\frac{d^2v}{dt^2}=-\frac{(2f'f''+2g'g'')\frac{dv}{dt}}{[(f')^2+(g')^2]^2}\Leftrightarrow\frac{d^2v}{dt^2}=-\frac{f'f''+g'g''}{(f')^2+(g')^2}\left(\frac{dv}{dt}\right)^2$$

thus ② holds. We can now conclude that all meridians which are parametrized by arc length are geodesics and hence has constant "energy" $|\gamma'(t)|^2$ along γ, because geodesic γ has constant speed $|\gamma'(t)|$.

No.

Date

2° We characterize the parallels $v = \text{const.}$, $u = u(s)$, parametrized by arc length, which are geodesics. ① gives $\frac{du}{dt} = \text{Const.}$, and ② becomes

$$\frac{ff'}{(f')^2+(g')^2}\left(\frac{du}{dt}\right)^2 = 0,$$ which holds only if $f' \equiv 0$ since $f \neq 0$ and $(f')^2+(g')^2 \neq 0$.

So a parallel is a geodesic only if $f' = 0$. This condition is also sufficient, since when it holds ② also holds automatically, and the fact that this parallel is parametrized by arc length gives $|\gamma'(t)| = 1$, i.e.

$$|\gamma'(t)|^2 = \left(-f(v)\sin u \frac{du}{dt}\right)^2 + \left(f(v)\cos u \frac{du}{dt}\right)^2 = f^2\left(\frac{du}{dt}\right)^2 = 1 \Rightarrow$$

$$2ff'\left(\frac{du}{dt}\right)^3 + f^2 \cdot 2\frac{du}{dt}\frac{d^2u}{dt^2} = 0 \Rightarrow 2f^2\frac{du}{dt}\frac{d^2u}{dt^2} = 0.$$ Note that $f \neq 0$,

and $\frac{du}{dt} \neq 0$ since $\frac{dv}{dt} = 0$ already and we don't want the parallel under consideration shrinks to a single point, thus $\frac{d^2u}{dt^2} = 0$ and ① holds.

We can now conclude that not all parallels are geodesics and thus the statement of the original problem. When a parallel is a geodesic, of course it will have constant "energy" $|\gamma'(t)|^2$; but this is generally not the case.

Now we return to the rest part of c). Consider a parallel passing through the point $\gamma(t) = (u(t), v(t))$. The tangent vector is given by

$$P'(t) = \left(-f(v_0)\cdot\sin u(t),\ f(v_0)\cdot\cos u(t),\ 0\right).$$ Then $|P'(t)| = |f(v_0)| = \text{Const.}$

$$\langle \gamma'(t), P'(t)\rangle = -f(v_0)\sin u(t)\left[f'(v_0)\frac{dv}{dt}\cos u(t) - f(v_0)\sin u(t)\frac{du}{dt}\right]$$
$$+ f(v_0)\cos u(t)\left[f'(v_0)\frac{dv}{dt}\sin u(t) + f(v_0)\cos u(t)\frac{du}{dt}\right]$$
$$= f(v_0)^2\frac{du}{dt}$$

Thus $$\cos\beta = \frac{f(v_0)^2\frac{du}{dt}}{|f|\,|P'(t)|} = \frac{|f(v(t))|\frac{du}{dt}}{\text{Const.}}$$

Note that $r = |f(v(t))|$, thus $r\cos\beta = \text{Const} \times (f(v(t)))^2\frac{du}{dt}$.

By ① we have $$\frac{dr}{dt} = \text{Const}\times\left[2ff'\frac{dv}{dt}\frac{du}{dt} + f^2\frac{d^2u}{dt^2}\right] = \text{Const}\cdot f^2\left(\frac{d^2u}{dt^2} + \frac{2ff'}{f^2}\frac{du}{dt}\frac{dv}{dt}\right) = 0.$$

Thus ①* implies the Clairaut's relation.

d) We first claim that any geodesic passing through the origin $(0,0,0)$ must be a meridian. Indeed, if $\gamma(t)$ passes through $(0,0,0)$, the plane through the origin containing the tangent vector along $\gamma(t)$ at the origin is a symmetric plane for the paraboloid. Since reflection along this plane is an isometry for the paraboloid, the uniqueness of the geodesic through $(0,0,0)$ with tangent vector along γ ensures that γ must lie in the plane. Thus γ is the intersection of the plane and the paraboloid, in other words γ must be the meridian. Hence a geodesic of the paraboloid which is not a meridian does not pass through $(0,0,0)$.

Now the paraboloid is given by:

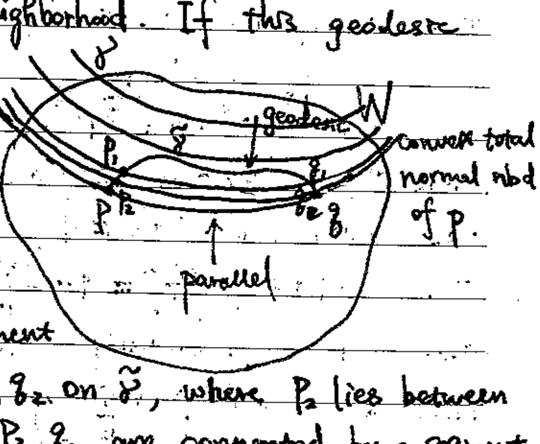
$$f(v) = v, \quad g(v) = v^2, \quad 0 < v < \infty, \quad \varepsilon < u < 2\pi + \varepsilon$$

We consider the Clairaut's relation:

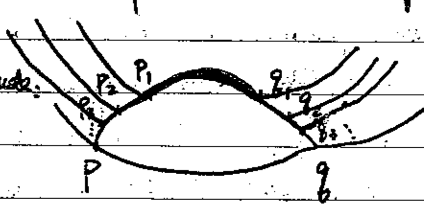
$$r \cos\beta = f(v)\cos\beta = v\cos\beta = v_0 = \text{const}.$$

thus either there exists t_0 such that $\beta(t_0) = 0$ or $\beta(t) \neq 0$ for all $t \in (-\infty, \infty)$.
Since locally r increases as β increases, if $\beta(t) \neq 0$ for all $t \in (-\infty, \infty)$ then $0 < \cos\beta < 1$, $v > v_0$, and v never attain the value v_0. This case implies that γ approaches asymptotically a parallel on the paraboloid.

• We claim that this is impossible unless the parallel itself is a geodesic on the paraboloid. In fact, consider a point p on this parallel, then there exists a convex total normal geodesic neighborhood around p. Choose $q \neq p$ in the intersection of the parallel with this neighborhood, then there is a geodesic $\tilde{\gamma}$ connecting p to q lying in this neighborhood. If this geodesic segment $\tilde{\gamma}$ is not part of the parallel, then since γ approaches this parallel, it must intersect with the geodesic segment $\tilde{\gamma}$ at two points, lying in W. Assume the two intersecting points are P_1, q_1. Then by the uniqueness of geodesics connecting P_1 to P_2 in W, the segment must lie in γ. Similarly, we can find P_2, q_2 on $\tilde{\gamma}$, where P_2 lies between P and P_1, q_2 lies between q and q_1, and P_2, q_2 are connected by a segment on γ. Again by the uniqueness of the geodesic in W, the segments connecting P_2 and q_2



on γ, $\tilde{\gamma}$ must coincide. Using the same argument, we can show that whenever γ intersects $\tilde{\gamma}$ in W it has to go through the segment connecting p_1 and q_1. Therefore γ does not approach the parallel between p and q, contradicting our assumption on γ.



There is one last case we need to exclude. What if the geodesic segment $\tilde{\gamma}$ does not intersect γ at all? By Clairaut's relation this is not going to happen at all: if β increases, r must increase, which means that $\tilde{\gamma}$ must "go up". Now we can conclude that if $\tilde{\gamma}$ does not lie in the parallel, there would be a contradiction on the hypothesis that γ approaches approximately the parallel. Therefore, if $\beta(t) \neq 0$ for all $t \in (-\infty, \infty)$, then γ approaches approximately a parallel which is itself a geodesic. However, according to our characterization in c), a parallel $v \equiv v_0 = \text{Const}$ is a geodesic only if $f'(v_0) = 0$. Note for a paraboloid $f'(v_0) = 1 \neq 0$ for all v_0. Thus on a paraboloid it is impossible that $\beta(t) \neq 0$ for all $t \in (-\infty, \infty)$.

- So we can now conclude that there exists t_0 such that $\beta(t_0) = 0$. First let us consider γ on $[t_0, +\infty)$. By Clairaut's relation, $v \cos\beta = v_0 = \text{const}$. Thus $v \geq v_0$ and $\cos\beta = v_0/v \Rightarrow \sin\beta = \sqrt{1 - v_0^2/v^2}$. Recall that in c) we established $\cos\beta = \dfrac{|f(v(t))| \frac{du}{dt}}{\text{Const}} \Rightarrow \dfrac{du}{dt} = \dfrac{\cos\beta}{|v|} \cdot 1 = \dfrac{v_0}{v^2}$ where we assumed the geodesic is parametrized by arc length and $v > 0$. Thus the winding number of γ along the z-axis is

$$n = \frac{1}{2\pi}\int_{t_0}^{\infty} \left|\frac{du}{dt}\right| dt = \frac{1}{2\pi}\int_{t_0}^{\infty} \frac{v_0}{v^2}\, dt = \frac{v_0}{2\pi}\int_{t_0}^{\infty} \frac{dt}{v^2}$$

Note that $\gamma(t) = (v(t)\cos u(t),\ v(t)\sin u(t),\ (v(t))^2)$ is parametrized by arc length, thus $|\gamma'(t)| = 1$, i.e. $(v(t))^2 \left(\frac{du}{dt}\right)^2 + (1^2 + 4(v(t))^2)\left(\frac{dv}{dt}\right)^2 = 1$, thus $(1+4v^2)\left(\frac{dv}{dt}\right)^2 = 1 - \frac{v_0^2}{v^2} \Rightarrow \frac{dv}{dt} = \frac{1}{\sqrt{1+4v^2}}\sqrt{1 - \frac{v_0^2}{v^2}}$. Note that $\frac{dv}{dt} \geq 0$, thus $v(t)$ is monotonically increasing as $t \to +\infty$, and hence $\sqrt{1 - \frac{v_0^2}{v^2}}$ is bounded from below away from 0 for all $t \geq t_0 + \eta$ for some $\eta > 0$. (When $t = t_0$, $\beta(t_0) = 0$ and thus $v = v_0$, $\sqrt{1 - \frac{v_0^2}{v^2}} = 0$. But we are interested in the behavior of $v(t)$ when t is very large). Write $\sqrt{1 - \frac{v_0^2}{v^2}} \geq M_0$. Then $\frac{dv}{dt} \geq \frac{M_0}{\sqrt{1+4v^2}} \Rightarrow \sqrt{1+4v^2}\, dv \geq M_0\, dt$

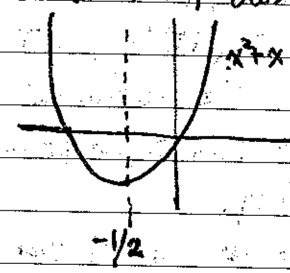
$$\Rightarrow \int_{v_0}^{v(t)} \sqrt{1+4v^2}\,dv \geq M_0\int_{t_0}^{t} dt \Rightarrow \int_{v_0}^{v(t)} \sqrt{1+4v+4v^2}\,dv \geq M_0(t-t_0)$$

$$\Rightarrow \int_{v_0}^{v(t)} (2v+1)\,dv \geq M_0(t-t_0) \Rightarrow \left[v^2+v\right]\Big|_{v_0}^{v(t)} \geq M_0(t-t_0)$$

$$\Rightarrow [v(t)]^2 + v(t) \geq M_0 t - M_0 t_0 + v_0^2 + v_0$$

$\Rightarrow$ when $t \to +\infty$, $[v(t)]^2 + v(t) \to +\infty \Rightarrow$ as $t \to +\infty$, $v(t) \to +\infty$.

Hence by a change of variable we have



$$n = \frac{v_0}{2\pi}\int_{t_0}^{+\infty} \frac{1}{v^2}\,dt = \frac{v_0}{2\pi}\int_{v_0}^{+\infty} \frac{1}{v^2}\left(\frac{dt}{dv}\right) dv$$

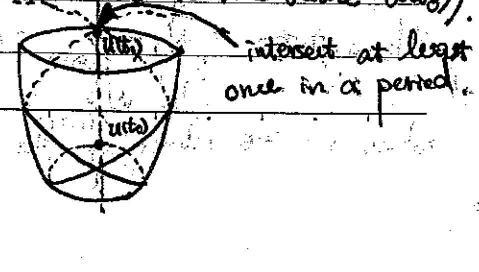
$$= \frac{v_0}{2\pi}\int_{v_0}^{+\infty} \frac{1}{v^2}\sqrt{\frac{v^2(1+4v^2)}{v^2-v_0^2}}\,dv$$

$$= \frac{v_0}{2\pi}\int_{v_0}^{+\infty} \frac{1}{v}\sqrt{\frac{1+4v^2}{v^2-v_0^2}}\,dv \geq \frac{v_0}{2\pi}\int_{v_0}^{+\infty} \frac{1}{v}\sqrt{\frac{4(v^2-v_0^2)}{v^2-v_0^2} + \frac{4v_0^2+1}{v^2-v_0^2}}\,dv$$

$$\geq \frac{v_0}{2\pi}\int_{v_0}^{+\infty} \frac{2}{v}\,dv = \frac{v_0}{\pi}\ln v\Big|_{v_0}^{+\infty} = +\infty$$

In words, as $t \to +\infty$ the geodesic $\gamma(t)$ revolve around the z-axis infinitely many times.

By Symmetry of the geodesic equations ① ②, $\gamma(-t)$ is also a geodesic. Applying the same argument, we know as $t \to -\infty$ the geodesic $\gamma(t)$ also revolves along the z-axis infinitely many times. Since t_0 is the point at which $\beta(t_0) = 0$, γ is tangent at a parallel at point $\gamma(t_0)$. Again by the Clairaut's relation, as $t \to -\infty$ the geodesic will also "go up" above the tangential parallel passing through $\gamma(t_0)$. Since both ends of $\gamma(t)$ goes up and revolve in different directions around the z-axis infinitely many times, they will intersect infinitely many times (at least once for every period, where a period for each end is defined as $u(t)$ returns to the value $u(t_0)$).



2. It is possible to introduce a Riemannian metric in the tangent bundle TM of a Riemannian manifold M in the following manner. Let $(p,v)\in TM$ and V, W be tangent vectors in TM at (p,v). Choose curves in TM

$$\alpha: t \longrightarrow (p(t), v(t)), \quad \beta: s \longrightarrow (q(s), w(s)),$$

with $p(0)=q(0)=p$, $v(0)=w(0)=v$, and $V=\alpha'(0)$, $W=\beta'(0)$.

Define an inner product on TM by

$$\langle V, W\rangle_{(p,v)} = \langle d\pi(V), d\pi(W)\rangle_p + \left\langle \frac{Dv}{dt}(0), \frac{Dw}{ds}(0)\right\rangle_p,$$

where $d\pi$ is the differential of $\pi: TM \longrightarrow M$.

a) Prove that this inner product is well-defined and introduces a Riemannian metric on TM.

b) A vector at $(p,v)\in TM$ that is orthogonal (for the metric above) to the fiber $\pi^{-1}(p)\approx T_pM$ is called a horizontal vector. A curve $t\longrightarrow (p(t), v(t))$ in TM is horizontal if its tangent vector is horizontal for all t. Prove that the curve $t\longrightarrow(p(t),v(t))$ is horizontal if and only if the vector field $v(t)$ is parallel along $p(t)$ in M.

c) Prove that the geodesic field is a horizontal vector field (i.e., it is horizontal at every point).

d) Prove that the trajectories of the geodesic field are geodesics on TM in the metric above.

Hint. Let $\bar\alpha = (\alpha(t), u(t))$ be a curve in TM. Show that $\ell(\bar\alpha)\geq \ell(\alpha)$ and that the inequality is verified if v is parallel along α. Consider a trajectory of the geodesic flow passing through (p,v) which is locally of the form $\bar\gamma(t) = (\gamma(t), \gamma'(t))$, where $\gamma(t)$ is a geodesic on M. Choose convex neighborhoods $W\subset TM$ of (p,v) and $V\subset M$ of p such that $\pi(W)=V$. Take two points $Q_1=(q_1,v_1)$, $Q_2=(q_2,v_2)$ in $\bar\gamma\cap W$. If $\bar\gamma$ is not a geodesic, there exists a curve $\bar\alpha$ in W passing through Q_1 and Q_2 such that $\ell(\bar\alpha)<\ell(\bar\gamma)=\ell(\gamma)$. Let $\alpha=\pi(\bar\alpha)$; since $\ell(\alpha)\leq\ell(\bar\alpha)$, this contradicts the fact that γ is a geodesic.

e) A vector at $(p,v)\in TM$ is called vertical if it is tangent to the fiber $\pi^{-1}(p)\approx T_pM$. Show that:

$$\langle W, W\rangle_{(p,v)} = \langle d\pi(W), d\pi(W)\rangle_p, \quad \text{if } W \text{ is horizontal},$$

$$\langle W, W\rangle_{(p,v)} = \langle W, W\rangle_p, \quad \text{if } W \text{ is vertical},$$

where we are identifying the tangent space to the fiber with T_pM.

- a) We choose a coordinate chart $\vec{x}: U \subset \mathbb{R}^n \longrightarrow M$ around $p \in M$. Then $\alpha(t) = (p_1(t), \cdots, p_n(t), v_1(t), \cdots, v_n(t))$, $\beta(s) = (q_1(s), \cdots, q_n(s), w_1(s), \cdots, w_n(s))$, with $V = (p_1'(0), \cdots, p_n'(0), v_1'(0), \cdots, v_n'(0))$, $W = (q_1'(0), \cdots, q_n'(0), w_1'(0), \cdots, w_n'(0))$.

To show the inner product defined in the problem is well-defined, assume we have another pair of curves $\tilde{\alpha}$, $\tilde{\beta}$ such that $\tilde{\alpha}(0) = \tilde{\beta}(0) = (p, v) = \alpha(0) = \beta(0)$, and $\tilde{\alpha}'(0) = \alpha'(0) = V$, $\tilde{\beta}'(0) = \beta'(0) = W$. Then

$$\frac{d}{dt}\Big|_{t=0} (\pi \circ \tilde{\alpha})(t) = \frac{d}{dt}\Big|_{t=0} (\tilde{p}_1(t), \cdots, \tilde{p}_n(t)) = (\tilde{p}_1'(0), \cdots, \tilde{p}_n'(0)) = (p_1'(0), \cdots, p_n'(0)) =$$

$$= \frac{d}{dt}\Big|_{t=0} (p_1(t), \cdots, p_n(t)) = \frac{d}{dt}\Big|_{t=0} (\pi \circ \alpha)(t)$$

Thus $d\pi(V)$ is independent of the choice of curve α or $\tilde{\alpha}$. Similarly $d\pi(W)$ is also well-defined, hence $\langle d\pi(V), d\pi(W) \rangle_p$ is independent of the choices of curves representing V and W.

Further, note that

$$\frac{D\tilde{v}}{dt}(0) = \nabla_{\tilde{p}'(t)} \tilde{v}(t) = \sum_j \tilde{p}_j' \nabla_j \Big(\sum_k \tilde{v}_k(t) \partial_k \Big) \Big|_{t=0} = \sum_j \tilde{p}_j'(0) \sum_k \Big(\partial_j \tilde{v}_k(p) \partial_k + \tilde{v}_m(0) \Gamma_{jm}^k \partial_k \Big)$$

$$= \sum_{j,k=1}^n \tilde{p}_j'(0) \Big(\partial_j \tilde{v}_k(p) + \tilde{v}_m(0) \Gamma_{jm}^k(p) \Big) \partial_k = \sum_{k=1}^n \Big(\tilde{v}_k'(0) + \sum_{j=1}^n \sum_{m=1}^n \tilde{p}_j'(0) \tilde{v}_m(0) \Gamma_{jm}^k(p) \Big) \partial_k$$

$$= \sum_{k=1}^n \Big(v_k'(0) + \sum_{j=1}^n \sum_{m=1}^n p_j'(0) v_m(0) \Gamma_{jm}^k(p) \Big) \partial_k = \frac{Dv}{dt}(0)$$

Similarly $\frac{D\tilde{w}}{ds}(0) = \frac{Dw}{ds}$. Thus $\left\langle \frac{Dv}{dt}(0), \frac{Dw}{ds}(0) \right\rangle_p$ is independent of the choices of curves representing V and W.

Combing the two points above, we know $\langle V, W \rangle_{(p,v)}$ is well-defined.

Now we show this defines a Riemannian metric on TM. The symmetry and bilinearity are obvious. To show the positive-definite-ness, first note that $\langle V, W \rangle_{(p,v)} \geq 0$ for all $V, W \in T_{(p,v)}(TM)$. If for some $V \in T_{(p,v)}(TM)$ there holds $\langle V, V \rangle_{(p,v)} = 0$, then $d\pi(V) = 0$ and $\frac{Dv}{dt}(0) = 0$, where $V = \alpha'(0)$ and $\alpha(t) = (p(t), v(t))$. This gives

$$0 = \frac{Dv}{dt}(0) = \sum_{k=1}^n \Big(v_k'(0) + \sum_{j=1}^n \sum_{m=1}^n p_j'(0) v_m(0) \Gamma_{jm}^k(p) \Big) \partial_k = \sum_{k=1}^n v_k'(0) \partial_k = 0$$

which implies $v_k'(0) = 0$ $\forall\, 1 \leq k \leq n$. Hence $p_k'(0) = v_k'(0) = 0$ for all $1 \leq k \leq n$, i.e. $V = \alpha'(0) = (p_1'(0), \cdots, p_n'(0), v_1'(0), \cdots, v_n'(0)) = 0$. This concludes the positive-definiteness and thus verifies that $\langle\, ,\, \rangle_{(p,v)}$ defined in the problem is a Riemannian connection.

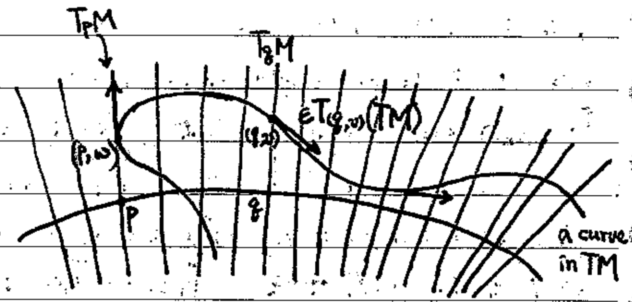
b) Solving this problem partly relies on how to understand elements in T_pM as elements in $T_{(p,v)}(TM)$.

In fact, for any $w \in T_pM$, consider a curve in TM defined as

$$\gamma(t) = (p,\ v + tw)$$

Then $\gamma(t)$ is a curve in TM.

The derivative of γ at $t=0$ is then



$$T_{(p,v)}(TM) \ni \frac{d}{dt}\Big|_{t=0} \gamma(t) = \gamma'(0) = (0, w)$$ which can be identified with $w \in T_pM$.

- If a curve $\alpha(t) = (p(t), v(t))$ is horizontal, then for each t_0 we have $\alpha'(t_0)$ orthogonal to any $w \in T_{p(t_0)}M \cong T_{(p(t_0),v(t_0))}(TM) = T_{\alpha(t_0)}(TM)$. Note that $d\pi(w) = \frac{d}{dt}\Big|_{t=t_0} (\pi\circ\gamma)(t) = \frac{d}{dt}\Big|_{t=t_0} Id_p = 0$

$$\frac{D(v+sw)}{ds}\Big|_{s=t_0} = \left[\frac{d}{ds}\Big|_{s=t_0}(v^i + sw^i) + \sum_{j,k}(v^j + sw^j)\frac{dp_k}{dt}\Gamma^i_{jk}\right]\partial_i = w^i\partial_i = w$$

where obviously we viewed $(p, v+sw)$ as a vector field along a constant curve Id_p. It follows that

$$0 = \langle \alpha'(t_0), \gamma'(t_0)\rangle = \langle d\pi(\alpha'(t_0)), d\pi(\gamma'(t_0))\rangle_p + \left\langle \frac{Dv}{dt}(t_0), \frac{Dw}{ds}(t_0)\right\rangle$$

$$= \left\langle \frac{Dv}{dt}(t_0), w\right\rangle_p.$$ Since $w \in T_pM$ is arbitrary, this implies that $\frac{Dv}{dt}(t_0) = 0$.

By the arbitrary choices of t_0, we know that $v(t)$ is parallel along $p(t)$ in M.

- Conversely, if $v(t)$ is parallel along $p(t)$ in M, $\frac{Dv}{dt}(t_0) = 0$ for each t_0, and we can reverse the whole argument above and conclude that $\alpha(t)$ is horizontal.

c) Recall a geodesic field on TM is a unique vector field on TM with trajectories of the form $t \longmapsto (\gamma(t), \gamma'(t))$ where γ is a geodesic on M. By b), it is a horizontal vector field if and only if $\gamma'(t)$ is parallel along $\gamma(t)$ in M, which it actually is since by definition $\nabla_{\gamma'(t)}\gamma'(t) = 0$ for all t on which the geodesic γ is defined.

d) Rather than computing the covariant derivatives in TM, we proceed as the hint suggests. Let $\bar{\alpha}(t) = (\alpha(t), v(t))$ be a curve in TM, then (assuming $t \in [a,b]$)

$$l(\bar{\alpha}) = \int_a^b |\bar{\alpha}'(t)|\, dt = \int_a^b \sqrt{|\alpha'(t)|^2 + \left|\frac{Dv}{dt}(t)\right|^2}\, dt \geqslant \int_a^b |\alpha'(t)|\, dt = l(\alpha)$$

where the equality holds if and only if $\frac{Dv}{dt} = 0$ for all $t \in (a,b)$, i.e. v is parallel along α.

Now let us consider a geodesic γ on M. Its geodesic field has trajectories $(\gamma(t), \gamma'(t))$ on TM. Assume $p = \gamma(t_0) \in M$, $v = \gamma'(t_0) \in T_pM$. Let V be a convex geodesic neighborhood of p in M, and let W be a convex geodesic neighborhood of (p,v) in TM satisfying $\pi(W) = V$ (note that this is possible because TM is locally a product manifold). Consider two arbitrary points $Q_1 = (q_1, v_1)$, $Q_2 = (q_2, v_2)$ in $\bar{\gamma} \cap W$, where $\bar{\gamma}(t) = (\gamma(t), \gamma'(t)) \subset TM$ is the trajectory of the geodesic field of γ. By the locally length minimizing property of geodesics in TM, if $\bar{\gamma}$ is not a geodesic in TM then there exists a curve $\bar{\alpha}$ in W passing through Q_1 and Q_2 such that $l(\bar{\alpha}) < l(\bar{\gamma}) = l(\gamma)$, where the last equality holds because γ is a geodesic in M. Then we have $l(\alpha) = l(\pi \circ \bar{\alpha}) \leq l(\bar{\alpha}) < l(\gamma)$, where $\alpha = \pi \circ \bar{\alpha}$ is a curve lying in V. This contradicts the locally length minimizing property of geodesics in M. Hence we conclude that $\bar{\gamma}$ is a geodesic on TM with the previously specified metric.

e) If W is horizontal, then by b) we know that $v(t)$ is parallel along $p(t)$ in M and thus $\frac{Dv}{dt} \equiv 0$, which gives $\langle W, W \rangle_{(p,v)} = \langle d\pi(W), d\pi(W) \rangle_p$.

If W is vertical, by definition it is tangent to the fiber $\pi^{-1}(p) \cong T_pM$, which means that there exists some path γ in T_pM with $\gamma(0) = v$ and $\gamma'(0) = W$. We can easily view this γ in T_pM as a path in TM by identifying $\gamma(t)$ with $(Id_p, \gamma(t))$. Then by the definition of covariant derivatives we have $\frac{D\gamma}{dt} \equiv W$, where the covariant derivative is understood along the constant curve Id_p. Moreover, we have $d\pi(W) = \frac{d}{dt}(\pi \circ (Id_p, \gamma(t))) = \frac{d}{dt} Id_p \equiv 0$. Therefore $\langle W, W \rangle_{(p,v)} = \left\langle \frac{D\gamma}{dt}(0), \frac{D\gamma}{dt}(0) \right\rangle_p = \langle W, W \rangle_p$. (We finally remark that a path in T_pM specifies a time-dependent tangent vector at $p \in M$, thus $\gamma(t)$ denotes a path in $I \times T_pM$ precisely as our previous notation $v(t)$ does. This identification has already been clarified at the beginning of the solution to b).)

No.

Date

3. Let G be a Lie group, $\mathfrak{g}$ its Lie algebra and let $X \in \mathfrak{g}$ (see Example 2.6, Chap. 1). The trajectories of X determine a mapping $\varphi: (-\varepsilon, \varepsilon) \longrightarrow G$ with $\varphi(0) = e$, $\varphi'(t) = X(\varphi(t))$.

a) Prove that $\varphi(t)$ is defined for all $t \in \mathbb{R}$ and that $\varphi(t+s) = \varphi(t) \cdot \varphi(s)$. ($\varphi: \mathbb{R} \rightarrow G$ is then called a <u>1-parameter subgroup</u> of G.)

<u>Hint:</u> Let $\varphi(t_0) = y$, $t_0 \in (-\varepsilon, \varepsilon)$. Show that, from the left invariance, $t \rightarrow y^{-1}\varphi(t)$, $t \in (-\varepsilon, \varepsilon)$, is also an integral curve of X passing through e for $t = t_0$. By uniqueness, $\varphi(t_0)^{-1}\varphi(t) = \varphi(t - t_0)$, hence φ can be extended out from t_0 in an interval of radius ε. This shows that $\varphi(t)$ is defined for all $t \in \mathbb{R}$. In addition $\varphi(t_0)^{-1} = \varphi(-t_0)$ and, since t_0 is arbitrary, we obtain $\varphi(t+s) = \varphi(t) \cdot \varphi(s)$.

b) Prove that if G has a bi-invariant metric $\langle\, , \rangle$ then the geodesics of G that start from e are 1-parameter subgroups of G.

<u>Hint:</u> Use the relation (see Eq. (9) of Chap. 2)

$$2\langle X, \nabla_Z Y\rangle = Z\langle X, Y\rangle + Y\langle X, Z\rangle - X\langle Y, Z\rangle + \langle Z, [X,Y]\rangle + \langle Y, [X,Z]\rangle - \langle X, [Y,Z]\rangle$$

and the fact that the metric is left invariant to prove that

$$\langle X, \nabla_Y Y\rangle = \langle Y, [X,Y]\rangle, \text{ where } X, Y \text{ and } Z \text{ are left invariant fields.}$$

Use also the fact that the bi-invariance of $\langle\, , \rangle$ implies that

$$\langle [U,X], V\rangle = -\langle U, [V,X]\rangle, \quad X, U, V \in \mathfrak{g}.$$

It follows that $\nabla_Y Y = 0$, for all $Y \in \mathfrak{g}$. Thus 1-parameter subgroups are geodesics. By uniqueness, geodesics are 1-parameter subgroups.

Proof: a) Let $t_0 \in (-\varepsilon, \varepsilon)$ and write $\varphi(t_0) = y$. It follows from the left invariance that

$$\frac{d}{dt}\left(y^{-1}\varphi(t)\right) = y^{-1}\frac{d\varphi(t)}{dt} = y^{-1}X(\varphi(t)) = X(y^{-1}\varphi(t))$$

which means that $t \longmapsto y^{-1}\varphi(t)$, $t \in (-\varepsilon, \varepsilon)$ is also an integral curve of X. This curve passes through e at $t = t_0$ by definition. The left-translated integral curve $t \longmapsto y^{-1}\varphi(t+t_0)$ then satisfies the same ODE which defines the integral curve $t \longmapsto \varphi(t)$, and the initial-values at $t=0$ also coincide. By the uniqueness of solution to an ODE with initial value, there holds $y^{-1}\varphi(t+t_0) = \varphi(t)$, or equivalently $\varphi(t+t_0) = \varphi(t_0)\varphi(t)$. Thus φ can be extended to interval $(-\varepsilon, \varepsilon+t_0)$ (if $t_0 > 0$) or $(-\varepsilon+t_0, \varepsilon)$ (if $t_0 < 0$). Since $\varepsilon > 0$ is fixed, we can extend

$\varphi(t)$ in this way to all $t \in \mathbb{R}$. Setting $t = -t_0 \in \mathbb{R}$ arbitrarily is then validified, which particularly gives $e = \varphi(0) = \varphi(t_0)\varphi(-t_0) \Rightarrow \varphi(-t_0) = \varphi(t_0)^{-1}$. Since t_0 can be chosen arbitrarily, we know $\varphi(t+s) = \varphi(t)\cdot\varphi(s)$ for all $t, s \in \mathbb{R}$, and $\varphi(t)^{-1} = \varphi(-t)$ for all $t \in \mathbb{R}$. Hence $\{\varphi(t) : t \in \mathbb{R}\}$ is a subgroup of G.

b) For $X, Y, Z \in \mathfrak{g}$ we have the formula

$$2\langle X, \nabla_Z Y\rangle = Z\langle X,Y\rangle + Y\langle X,Z\rangle - X\langle Y,Z\rangle + \langle Z,[X,Y]\rangle + \langle Y,[X,Z]\rangle - \langle X,[Y,Z]\rangle. \quad (*)$$

The left invariance of the metric gives

$$\langle X(g), Y(g)\rangle_g = \langle (L_{g^{-1}})_*(X(g)), (L_{g^{-1}})_*(Y(g))\rangle_e = \langle X(e), Y(e)\rangle_e \qquad \forall g \in G$$

$$\langle Y(g), Y(g)\rangle_g = \langle (L_{g^{-1}})_*(Y(g)), (L_{g^{-1}})_*(Y(g))\rangle_e = \langle Y(e), Y(e)\rangle_e \qquad \forall g \in G$$

which means $\langle X,Y\rangle$ and $\langle Y,Y\rangle$ are both constants. Thus $Y\langle X,Y\rangle = 0 = X\langle Y,Y\rangle$.

Letting $Z = Y$ in $(*)$, we get

$$2\langle X, \nabla_Y Y\rangle = \cancel{Y\langle X,Y\rangle} + \cancel{Y\langle X,Y\rangle} - \cancel{X\langle Y,Y\rangle} + 2\langle Y,[X,Y]\rangle - \cancel{\langle X,[Y,Y]\rangle} = 2\langle Y,[X,Y]\rangle,$$

i.e. $\langle X, \nabla_Y Y\rangle = \langle Y,[X,Y]\rangle$. Moreover, since the metric is bi-invariant, there holds

$$\langle [U,X], V\rangle = -\langle U, [V,X]\rangle \quad \text{for all } X, U, V \in \mathfrak{g}.$$

Thus $\langle Y,[X,Y]\rangle = \langle [X,Y], Y\rangle = -\langle [Y,X], Y\rangle = \langle Y, [Y,X]\rangle = -\langle Y,[X,Y]\rangle$ and hence $\langle Y,[X,Y]\rangle = 0$ for all $X, Y \in \mathfrak{g}$. It follows that

$$\langle X, \nabla_Y Y\rangle = \langle Y,[X,Y]\rangle \quad \text{for all } X, Y \in \mathfrak{g}$$

and thus $\nabla_Y Y = 0$ for all $Y \in \mathfrak{g}$.

Now for any 1-parameter subgroup $\varphi(t)$ of G, let $Y(\varphi(t))$ denote for the vector field associated to $\varphi(t)$, i.e. $\varphi'(t) = Y(\varphi(t))$. Then we see that

$$\nabla_{\varphi'(t)}\varphi'(t) = \nabla_{Y(\varphi(t))} Y(\varphi(t)) = 0 \qquad \text{since } Y(\varphi(t)) \in \mathfrak{g}.$$

Hence 1-parameter subgroups are geodesics. Conversely, given an arbitrary $v \in \mathfrak{g}$, there is a unique 1-parameter subgroup $\varphi(t)$ of G satisfying $\varphi(0) = e$ and $\varphi'(0) = v$. Since we just proved this 1-parameter subgroup is a geodesic, by the uniqueness of geodesics in a geodesic normal neighborhood around e, we can conclude that under the bi-invariant metric the geodesics of G that start from e are 1-parameter subgroups of G.

No.

Date

4. A subset A of a differentiable manifold M is contractible to a point $a \in A$ when the mapping id_A (identity on A) and $k_a: x \in A \longrightarrow a \in A$ are homotopic (with base point a). A is contractible if it is contractible to one of its points.

a) Show that a convex neighborhood in a Riemannian manifold M is a contractible subset (with respect to any of its points).

b) Let M be a differentiable manifold. Show that there exists a covering $\{\mathcal{U}_\alpha\}$ of M with the following properties:

i) $\mathcal{U}_\alpha$ is open and contractible, for each α.

ii) If $\mathcal{U}_{\alpha_1}, \dots, \mathcal{U}_{\alpha_r}$ are elements of the covering, then $\bigcap_{i=1}^{r} \mathcal{U}_{\alpha_i}$ is contractible.

Proof: a). Fix an arbitrary point $a \in M$; and let A be a convex total normal neighborhood of a in M. Then for any $x \in A$ there exists a geodesic $\gamma_x: I \longrightarrow M$ such that $\gamma_x(0) = a$ and $\gamma_x(1) = x$. Define a continuous mapping $F: I \times A \longrightarrow A$ by $F(s, x) = \gamma_x(1-s)$. Since A is convex, F is well defined, i.e. the image of F is contained in A. Moreover, it is easy to see that F is continuous, with $F(0,x) = \gamma_x(1) = x = id_A(x)$ and $F(1,x) = \gamma_x(0) = a = k_a(x)$. Thus F is a homotopy from id_A to k_a, i.e. id_A and k_a are homotopic with base point a. Since A is a total normal neighborhood of $a \in M$, we can replace a with any $a' \in A$ and the argument above still holds.

b) For any $x \in M$ choose a convex neighborhood $\mathcal{U}_x$. Since M is second countable, it is Lindelöf, i.e. every open cover of M has a countable subcover. Hence the covering $\{\mathcal{U}_\alpha\}$ of M exists for some countable index set Λ. Each $\mathcal{U}_\alpha$ is a convex neighborhood of some $x_\alpha \in M$ and thus open and contractible. Moreover, since finite intersections of convex neighborhood is still a convex neighborhood (if nonempty), thus $\bigcap_{i=1}^{r} \mathcal{U}_{\alpha_i}$ is contractible for each finite subcollection of neighborhoods in the covering $\{\mathcal{U}_\alpha\}$.

5. Let M be a Riemannian manifold and $X \in \mathcal{X}(M)$. Let $p \in M$ and let $\mathcal{U} \subset M$ be a neighborhood of p. Let $\varphi: (-\varepsilon, \varepsilon) \times \mathcal{U} \longrightarrow M$ be a differentiable mapping such that for any $q \in \mathcal{U}$ the curve $t \longrightarrow \varphi(t, q)$ is a trajectory of X passing through q at $t=0$ ($\mathcal{U}$ and φ are given by the fundamental theorem for ordinary differential equations; Cf. Theorem 2.2). X is called a Killing field (or an infinitesimal isometry) if, for each $t_0 \in (-\varepsilon, \varepsilon)$, the mapping $\varphi(t_0, \ ): \mathcal{U} \subset M \longrightarrow M$ is an isometry.

Prove that:

a) A vector field v on $\mathbb{R}^n$ may be seen as a map $v: \mathbb{R}^n \longrightarrow \mathbb{R}^n$; we say that the field is linear if v is a linear map. A linear field on $\mathbb{R}^n$, defined by a matrix A, is a Killing field if and only if A is anti-symmetric.

b) Let X be a Killing field on M, $p \in M$, and let $\mathcal{U}$ be a normal neighborhood of p on M. Assume that p is a unique point of $\mathcal{U}$ that satisfies $X(p)=0$. Then, in $\mathcal{U}$, X is tangent to the geodesic spheres centered at p.

c) Let X be a differentiable vector field on M and let $f: M \longrightarrow N$ be an isometry. Let Y be a vector field on N defined by $Y(f(p)) = df_p(X(p))$, $p \in M$. Then Y is a Killing field if and only if X is also a Killing field.

d) X is Killing $\Leftrightarrow \langle \nabla_Y X, Z \rangle + \langle \nabla_Z X, Y \rangle = 0$ for all $Y, Z \in \mathcal{X}(M)$ (the equation above is called the Killing equation).

Hint for $\Rightarrow$: By continuity, it suffices to prove the equation above for points $q \in \mathcal{U}$ where $X(q) \neq 0$. If this is the case, let $S \subset \mathcal{U}$ be a submanifold of $\mathcal{U}$, passing through q, normal to $X(q) \neq 0$ at q, with $\dim S = \dim M - 1$. Let $(x_1, \cdots, x_{n-1})$ be coordinates in a neighborhood $V \subset S$ of q such that $(x_1, \cdots, x_{n-1}, t)$ are coordinates in a neighborhood $V \times (-\varepsilon, \varepsilon) \subset \mathcal{U}$ and $X = \frac{\partial}{\partial t}$. Putting $X_i = \frac{\partial}{\partial x_i}$, obtain

$$\langle \nabla_{X_j} X, X_i \rangle + \langle \nabla_{X_i} X, X_j \rangle = X\langle X_i, X_j \rangle - \langle [X, X_i], X_j \rangle - \langle [X, X_j], X_i \rangle = \frac{\partial}{\partial t}\langle X_i, X_j \rangle = 0,$$

where in the last equality the fact was used that X is a Killing field.

e) Let X be a Killing field on M with $X(q) \neq 0$, $q \in M$. Then there exists a system of coordinates $(x_1, \cdots, x_n)$ in a neighborhood of q, so that the coefficients g_{ij} of the metric in this system of coordinates do not depend on x_n.

No.

Date

- Proof: a) The isometries of $\mathbb{R}^n$ are elements in $O(n)$. For fixed $q \in \mathbb{R}^n$, the solution to the ODE with initial value $\begin{cases} \frac{\partial}{\partial t}\varphi(t,q) = A\varphi(t,q) \\ \varphi(0,q) = q \end{cases}$ is given by $\varphi(t,q) = (\exp(tA))\, q$ uniquely.

Then for fixed t, $q \longmapsto \varphi(t,q)$ is an isometry $\Leftrightarrow \exp(tA)[\exp(tA)]^T = I$
$\Leftrightarrow \exp(tA)\exp(tA^T) = I \Leftrightarrow \exp(t(A+A^T)) = I$.

If $A+A^T = 0$, i.e. A is anti-symmetric, then obviously $\exp(t(A+A^T)) = \exp 0 = I$.
Conversely, if $\exp(t(A+A^T)) = I$, then differentiating at $t=0$ gives $A+A^T=0$.

b) The most important but trivial observation is $\varphi(t,p) = p$ for all $t \in (-\varepsilon, \varepsilon)$.
Note that the integral curve of X passing through p at $t=0$ satisfies

$$\begin{cases} \frac{\partial}{\partial t}\varphi(t,p) = X(\varphi(t,p)) & (1) \\ \varphi(0,p) = p & (2) \end{cases}$$

and obviously $\varphi(t,p) = p$ satisfies both (1) and (2) since $X(p) = 0$.
By the uniqueness of the ODE initial value problem, we get $\varphi(t,p) = p$ for all $t \in (-\varepsilon, \varepsilon)$.

The reason that we want p to be an isolated zero point of X is because we want to show a non-trivial $X(q)$ is tangent to the geodesic sphere centered at p, at each point $q \in \partial B_a(p) = S_a(p)$.

Let $q = \exp_p(v) \in S_a(p)$, where $|v| = a$. $\gamma: t \longmapsto \exp(tv)$ is then a geodesic connecting p to q when t varies from 0 to 1. Since $q \longmapsto \varphi(s,q)$ is an isometry for any fixed $s \in (-\varepsilon,\varepsilon)$, the image γ_s of γ under this isometry is still a geodesic, and the speed also remains the same. Since $\mathcal{U}$ is a normal neighborhood, by the uniqueness of geodesics in $\mathcal{U}$ we conclude that $\gamma_s(t) = \varphi(s, \exp_p(tv)) = \exp_p(t\,v(s))$ for some $v(s) \in T_pM$, $|v(s)| = 1$. As s varies in $(-\varepsilon,\varepsilon)$, $v(s)$ is a curve in T_pM with $v(0)=v$. (In fact, for each fixed s one can determine $v(s)$ by $v(s) = \frac{\partial}{\partial t}\big|_{t=0}\gamma_s(t)$.) Thus $\gamma_s(1) \in S_a(p)$ for all s, $\gamma_0(1)=q$.

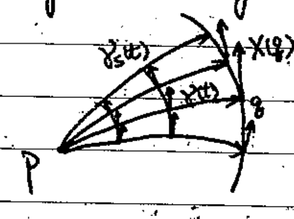
Note that $\left.\frac{\partial \gamma_s(t)}{\partial t}\right|_{\substack{t=1\\s=0}} = (d\exp_p)_v v$, $\left.\frac{\partial \gamma_s(t)}{\partial s}\right|_{\substack{t=1\\s=0}} = (d\exp_p)_v v'(0)$.

Since $|v(s)| \equiv a$ for $s \in (-\varepsilon,\varepsilon)$, we know $\langle v'(0), v(0)\rangle_p = 0$. Hence by the Gauss Lemma, $\langle (d\exp_p)_v v, (d\exp_p)_v v'(0)\rangle_{T_qM} = \langle v, v'(0)\rangle_{T_pM} = 0$. Moreover, note that

$$\left.\frac{\partial \gamma_s(t)}{\partial s}\right|_{\substack{t=1\\ s=0}} = \left.\frac{\partial}{\partial s}\varphi(s, \exp_p(tv))\right|_{\substack{t=1\\ s=0}} = \left.X(\varphi(s,\exp_p(tv)))\right|_{\substack{t=1\\ s=0}} = X(\exp_p(v)) = X(q).$$

$$\left.\frac{\partial \gamma_s(t)}{\partial t}\right|_{\substack{t=1\\ s=0}} = \gamma_0'(1) = \gamma'(1)$$

thus $\langle \gamma'(1), X(q)\rangle_q = 0$, i.e. X is tangent to the geodesic spheres centered at p.



Remark: $\gamma_s(t)$ is like a variation with fixed starting point p.

c) We claim that $\psi(t, f(p)) := f(\varphi(t,p))$ is the local flow associated to Y.

In fact, $\psi(0, f(p)) = f(\varphi(0,p)) = f(p)$

$$\forall\, t_0 \in (-\varepsilon, \varepsilon) \quad \left.\frac{\partial}{\partial t}\right|_{t=t_0}\psi(t, f(p)) = \left.\frac{\partial}{\partial t}\right|_{t=t_0} f(\varphi(t,p)) = \left.\frac{\partial}{\partial t}\right|_{t=t_0} f[\varphi(t-t_0, \varphi(t_0,p))] =$$

$$= df_{\varphi(t_0,p)}\left(\left.\frac{\partial}{\partial t}\right|_{t=t_0}\varphi(t-t_0, \varphi(t_0,p))\right) = df_{\varphi(t_0,p)}\big(X(\varphi(t_0,p))\big) = Y(f(\varphi(t_0,p)))$$

$$= Y(\psi(t_0, f(p))) \quad \square$$

In other words, for each $q \in N$ one has

$$\begin{cases} \frac{\partial}{\partial t}\psi(t,q) = Y(\psi(t,q)) \\ \psi(0,q) = 0 \end{cases}$$

by the uniqueness of local flows for smooth vector fields, the claim is verified.

Now note that $\psi(t, f(p)) = f(\varphi(t,p))$ implies $d\psi^t_{f(p)} \circ df_p = df_{\varphi^t(p)} \circ d\varphi^t_p$, or equivalently $d\psi^t_{f(p)} = df_{\varphi^t(p)} \circ d\varphi^t_p \circ (df_p)^{-1}$, thus for any $u, v \in T_{f(p)}N$ we have

$$\langle d\psi^t_{f(p)}(u), d\psi^t_{f(p)}(v)\rangle_{f(p)} = \left\langle \left[df_{\varphi^t(p)} \circ d\varphi^t_p \circ (df_p)^{-1}\right](u), \left[df_{\varphi^t(p)} \circ d\varphi^t_p \circ (df_p)^{-1}\right](v)\right\rangle_{f(p)}$$

$$\overset{(*)}{=\!=} \left\langle d\varphi^t_p(df_p^{-1}(u)), d\varphi^t_p(df_p^{-1}(v))\right\rangle_p$$

where at $(*)$ we used the assumption that f is an isometry. Recall that $\langle df_p^{-1}(u), df_p^{-1}(v)\rangle_p = \langle u, v\rangle_{f(p)}$, we can now conclude that Y is a Killing field

$\Leftrightarrow \langle u,v\rangle_{f(p)} = \langle d\psi^t_{f(p)}(u), d\psi^t_{f(p)}(v)\rangle_{f(p)} \Leftrightarrow \langle df_p^{-1}(u), df_p^{-1}(v)\rangle_p = \langle d\varphi^t_p(df_p^{-1}(u)), d\varphi^t_p(df_p^{-1}(v))\rangle_p$

$\Leftrightarrow$ X is a Killing field.

No.

Date

d) We can use do Carmo's hint, but here is a cleaner proof communicated by Prof. Lenhard Ng.

Lemma $\left.\frac{d}{dt}\right|_{t=0}\left(\left(\varphi_t^* g\right)(Y,Z)\right)(p) = g(\nabla_Y X, Z)(p) + g(\nabla_Z X, Y)(p)$

where $X, Y, Z \in \mathrm{Vect}(M)$ and φ_t = time-t flow for X.

Remark: this is the Lie derivative $\mathcal{L}_X g(Y,Z)$.

Proof of the lemma. $(\varphi_t^* g)(Y,Z)(p) = g(\varphi_{t*}Y, \varphi_{t*}Z)(\varphi_t(p)) = h_t \circ \varphi_t(p)$ where $h_t: M \to \mathbb{R}$ is given by $h_t(p) = g(\varphi_{t*}Y_{\varphi_t^{-1}(p)}, \varphi_{t*}Z_{\varphi_t^{-1}(p)})$. In particular, $h_0(p) = g(Y_p, Z_p)$. Now by definition

$$\left.\frac{d}{dt}\right|_{t=0} h_t(p) = g(-\mathcal{L}_X Y, Z)(p) + g(Y, -\mathcal{L}_X Z)(p) = g(-[X,Y], Z)(p) + g(Y, -[X,Z])(p),$$

and $\left.\frac{d}{dt}\right|_{t=0} \varphi_t(p) = X(p)$, so

$$\left.\frac{d}{dt}\right|_{t=0} (\varphi_t^* g)(Y,Z)(p) = \left.\frac{d}{dt}\right|_{t=0} (h_t \circ \varphi_t)(p) = g(-[X,Y], Z) + g(Y, -[X,Z]) + Xg(Y,Z)$$

$$= g(-\nabla_X Y + \nabla_Y X, Z) + g(Y, -\nabla_X Z + \nabla_Z X) + g(\nabla_X Y, Z) + g(Y, \nabla_X Z)$$

$$= g(\nabla_Y X, Z) + g(Y, \nabla_Z X). \quad \blacksquare$$

If X is Killing, then $\left.\frac{d}{dt}\right|_{t=0}\left((\varphi_t^* g)(Y,Z)\right)(p) = 0$, by the previous lemma we have $g(\nabla_Y X, Z) + g(Y, \nabla_Z X) = 0$.

Conversely, since $\varphi_{t+t'} = \varphi_t \circ \varphi_{t'}$, the lemma implies that

$$\frac{d}{dt}\left((\varphi_t^* g)(Y,Z)\right)(p) = g(\nabla_{\varphi_{t*}Y} X, \varphi_{t*}Z) + g(\varphi_{t*}Y, \nabla_{\varphi_{t*}Z} X)$$

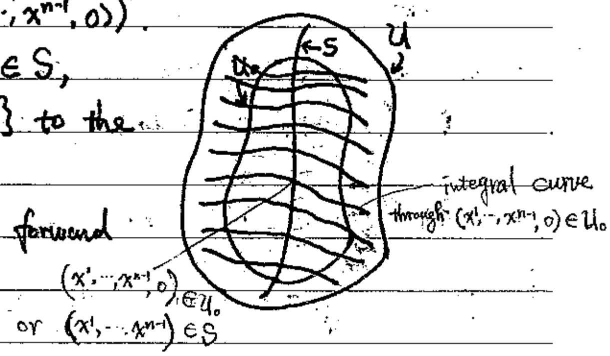
for arbitrary t. So if $g(\nabla_Y X, Z) + g(Y, \nabla_Z X) = 0$ for all $Y, Z \in \mathrm{Vect}(M)$ then $\frac{d}{dt}\left((\varphi_t^* g)(Y,Z)\right) \equiv 0$. Thus $(\varphi_t^* g)(Y,Z) = g(Y,Z)$ and X is Killing.

e) First we show that, for any smooth vector field X on M, if $X(q) \neq 0$ for some $q \in M$, then q has a neighborhood on which there exists a system of coordinates in which X has coordinate expression $\partial/\partial x_n$. This is known as "Canonical Form for a Regular Vector Field" (cf. John Lee, "Introduction to Smooth Manifolds", Theorem 13.5, pp. 331).

Let us begin with choosing any coordinates (x^i) on a neighborhood $\mathcal{U}$ of q, with q corresponding to 0. By composing with a linear transformation, we may assume without loss of generality that $X(q) = \frac{\partial}{\partial x^n}\Big|_q$. This is possible since we assumed $X(q) \neq 0$. Let $\varphi : (-\varepsilon,\varepsilon) \times \mathcal{U}_0 \longrightarrow M$ be the local flow of X, where $\mathcal{U}_0 \subset \mathcal{U}$ is chosen such that the image of $(-\varepsilon,\varepsilon)\times\mathcal{U}_0$ under the flow φ falls inside $\mathcal{U}$. Let $S \subset \mathbb{R}^{n-1}$ be the set $S = \{(x^1,\cdots,x^{n-1}) : (x^1,\cdots,x^{n-1},0) \in \mathcal{U}_0\}$, and define a smooth map $\psi : (-\varepsilon,\varepsilon)\times S \longrightarrow \mathcal{U}$ by

$$\psi(x^1,\cdots,x^{n-1},t) = \varphi(t,(x^1,\cdots,x^{n-1},0)).$$

Geometrically, for each fixed $(x^1,\cdots,x^{n-1}) \in S$, ψ maps the interval $(-\varepsilon,\varepsilon)\times\{(x^1,\cdots,x^{n-1})\}$ to the "integral" curve through $(x^1,\cdots,x^{n-1},0)$.



Under this smooth map ψ, "$\partial/\partial t$" is pushed forward to X. In fact,

$$\left(\psi_*\frac{\partial}{\partial t}\Big|_{(t_0,x_0)}\right) f = \frac{\partial}{\partial t}\Big|_{(t_0,x_0)} (f\circ\psi)$$

$$= \frac{\partial}{\partial t}\Big|_{t=t_0} f\big(\varphi(t,(x_0,0))\big) = X\big(\varphi(t_0,(x_0,0))\big) f = X\big(\psi(t_0,x_0)\big) f$$

where $x_0 = (x^1,\cdots,x^{n-1})$ is a point on S. Moreover, when restricted to $\{0\}\times S$; $\psi(x^1,\cdots,x^{n-1},0) = \varphi(0,(x^1,\cdots,x^{n-1},0)) = (x^1,\cdots,x^{n-1},0)$, thus for $1 \le i \le n-1$

$$\left(\psi_*\frac{\partial}{\partial x^i}\Big|_{(0,0)}\right) f = \frac{\partial}{\partial x^i}\Big|_{(0,0)} (f\circ\psi) = \frac{\partial}{\partial x^i}\Big|_{(0,0)} f(\psi(x^1,\cdots,x^{n-1},0)) = \frac{\partial}{\partial x^i}\Big|_{(0,0)} f(x^1,\cdots,x^{n-1},0)$$

$$= \frac{\partial}{\partial x^i}\Big|_{(0,0)} f \qquad \text{i.e.} \quad \psi_*\frac{\partial}{\partial x^i}\Big|_{(0,0)} = \frac{\partial}{\partial x^i}\Big|_{(0,0)} \quad 1 \le i \le n-1$$

$$\psi_*\frac{\partial}{\partial t}\Big|_{(0,0)} = X(q) = \frac{\partial}{\partial x^n}\Big|_{(0,0)}.$$

It follows that $\psi_* : T_{(0,0)}\big((-\varepsilon,\varepsilon)\times S\big) \longrightarrow T_q M$ is an isomorphism. Thus by the inverse function theorem we can pick neighborhoods W of $(0,0)$ in $(-\varepsilon,\varepsilon)\times S$ and Y of q such that $\psi : W \to Y$ is a diffeomorphism. Choosing (ψ, W) as a coordinate chart around $q \in M$, we easily see that $X = \partial/\partial t$. Renaming t to x^n finishes the whole construction.

[Remark: This argument also validifies the hint in do Carmo for problem d).]

No.

Date

Now we claim that under the coordinate chart constructed above the coefficients g_{ij} of the metric do not depend on x_n. In fact, note that

$$0 = \mathcal{L}_X g = \mathcal{L}_X(g_{ij}\,dx^i \otimes dx^j) = (X g_{ij})\,dx^i \otimes dx^j + g_{ij}(\mathcal{L}_X dx^i) \otimes dx^j + g_{ij}\,dx^i \otimes (\mathcal{L}_X dx^j)$$

$$= \frac{\partial g_{ij}}{\partial x^n} dx^i \otimes dx^j + g_{ij}\, d(\mathcal{L}_X x^i) \otimes dx^j + g_{ij}\, dx^i \otimes d(\mathcal{L}_X x^j)$$

$$= \frac{\partial g_{ij}}{\partial x^n} dx^i \otimes dx^j + g_{ij}\, d\left(\frac{\partial x^i}{\partial x^n}\right) \otimes dx^j + g_{ij}\, dx^i \otimes d\left(\frac{\partial x^j}{\partial x^n}\right)$$

$$= \frac{\partial g_{ij}}{\partial x^n} dx^i \otimes dx^j + g_{ij}\, d\delta_n^i \otimes dx^j + g_{ij}\, dx^i \otimes d\delta_n^j = \frac{\partial g_{ij}}{\partial x^n} dx^i \otimes dx^j$$

where we used properties of the Lie derivative:

① $\mathcal{L}_X(df) = d(\mathcal{L}_X f)$ for $f \in C^\infty(M)$ (see Lee, smooth manifold Corollary 13.13, pp. 340)

② $\mathcal{L}_X f = Xf$ for $f \in C^\infty(M)$ (see Lee, smooth manifold, Proposition 13.11, pp. 339).

Therefore $\frac{\partial g_{ij}}{\partial x^n} \equiv 0$ for $1 \le i, j \le n$, i.e. g_{ij} does not depend on x^n.

6. Let X be a Killing field (Cf. Exercise 5) on a connected Riemannian manifold M. Assume that there exists a point $q \in M$ such that $X(q) = 0$ and $\nabla_Y X(q) = 0$, for all $Y(q) \in T_qM$. Prove that $X \equiv 0$.

Hint: Show that, for all t, the local isometry $\varphi(t, \ ) : U \subset M \longrightarrow M$ generated by the field X (Cf. Exercise 5) leaves the point q fixed and its differential at q, as a linear map of T_qM, is the identity. For this, observe that $d\varphi_t : T_qM \longrightarrow T_qM$ for all t. In addition,

$$[X,Y](q) = (\nabla_X Y - \nabla_Y X)(q) = 0, \text{ by the hypothesis.}$$

Since $0 = [Y,X](q) = \lim_{t\to 0} \frac{1}{t}[d\varphi_t - Id](Y) = \frac{d}{dt}(d\varphi_t)\Big|_{t=0}$ and $d\varphi_{s+t} = d\varphi_s \cdot d\varphi_t$, conclude that $d\varphi_t$ does not depend on t, and it is equal to Id. Now use the exponential map to show that such an isometry is the identity on M.

- Proof: We proceed as the hint suggests. That $\varphi(t, q) = q$ for any $t \in (-\varepsilon, \varepsilon)$ is already established in b) of last problem (Exercise 5, Chapter 3). Thus $d\varphi_t$ is from T_qM to T_qM. Note that $(\nabla_X Y)(q) = 0$ for any $Y(q) \in T_qM$ since $\nabla_X Y$ is linear in the variable X. Moreover, by assumption $(\nabla_Y X)(q) = 0$ for all $Y(q) \in T_qM$, thus

$[X,Y](q) = (\nabla_X Y - \nabla_Y X)(q) = 0$. Note that $[X,Y](q) = \lim_{t\to 0}\frac{1}{t}[Y - d\varphi_t(Y)](\varphi_t(p))$, thus $0 = [Y,X](q) = \lim_{t\to 0}\frac{1}{t}[d\varphi_t(Y(q)) - Y(q)] = \left(\frac{d}{dt}\Big|_{t=0} d\varphi_t\right)Y(q)$. By the arbitrariness of $Y(q)\in T_qM$, we know $\frac{d}{dt}\Big|_{t=0} d\varphi_t = 0$. Note that $\varphi_{s+t} = \varphi_s\cdot\varphi_t$, we have $d\varphi_{s+t} = d\varphi_s\cdot d\varphi_t$, which gives $\frac{d}{dt}\Big|_{t=t_0} d\varphi_t = 0$ for any $t_0\in(-\varepsilon,\varepsilon)$. Thus $d\varphi_t$ does not depend on t, and hence $d\varphi_t \equiv d\varphi_0 = d(Id) = Id$.

Now recall the lemma we proved in the solution to Exercise 5 of Chapter 1, which says a Riemannian isometry is uniquely defined by the image at one point together with its differential at the same point. Therefore, $\varphi_t(p) = p$ and $d\varphi_t(p) = Id$ implies $\varphi_t = Id$, which further gives $X\equiv 0$.

7. (Geodesic frame). Let M be a Riemannian manifold of dimension n and let $p\in M$. Show that there exists a neighborhood $U\subset M$ of p and n vector fields $E_1,\cdots,E_n\in\mathfrak{X}(U)$, orthonormal at each point of U, such that, at p, $\nabla_{E_i}E_j(p) = 0$. Such a family E_i, $i=1,\cdots,n$, of vector fields is called a (local) geodesic frame at p.

- Proof: Choose an orthonormal basis $e_1,\cdots,e_n$ for $T_pM\cong\mathbb{R}^n$, and let $x^1,\cdots,x^n$ be the associated coordinates on T_pM such that $e_i = \partial_i = \partial/\partial x^i$ for $1\le i\le n$. For each point q in a normal ball U around p, $q = \exp_p v$, we can define $E_i(q) = (d\exp_p)_v\,\partial_i$, $1\le i\le n$. Since $(d\exp_p)_0 = Id$, we know that $\{E_1(p),\cdots,E_n(p)\}$ is an orthonormal basis for T_pM. Using $\exp_p$, one can pull back the metric on $U\subset M$ to a metric on T_pM (or more precisely $B_r(0)\subset T_pM$, but then one can extend it to the whole of T_pM). Then $\exp_p$ is an isometry from $B_r(0)\subset T_pM$ to U. Under this pull-back map, a geodesic through p lying in U is pulled back to a straight line in T_pM, passing through $0\in T_pM$. Since isometry preserves geodesics, $t\mapsto(v^i\partial_i)t$ is a geodesic in T_pM under the pull-back metric, for any fixed $v^1,\cdots,v^n\in\mathbb{R}$. Thus $x^i = v^i t$ satisfies the geodesic equations $\frac{d^2x^k}{dt^2} + \Gamma^k_{ij}\frac{dx^i}{dt}\frac{dx^j}{dt} = 0$. But $d^2x^k/dt^2\big|_{t=0} = 0$, $dx^i/dt\big|_{t=0} = v^i$, thus $\Gamma^k_{ij}(0)v^iv^j = 0$. By the arbitrariness of $\{v^i\}_{i=1}^n$, we have $\Gamma^k_{ij}(0) = 0$ for all $1\le i,j,k\le n$.

No.

Date

It follows that $(\nabla_i \partial_j)(0) = 0$ for any $1 \le i, j \le n$. Pushing forward everything to $\mathcal{U}$, one obtain $(\nabla_{E_i} E_j)(p) = 0$ for all $1 \le i, j \le n$.
Now we have vector fields $E_1, \dots, E_n$ with $\{E_1(p), \dots, E_n(p)\}$ orthonormal and $(\nabla_{E_i} E_j)(p) = 0$ $\forall\ 1 \le i, j \le n$. Using a Gram-Schmidt process, we can make $E_1, \dots, E_n$ orthonormal at each point in $\mathcal{U}$. We claim that this does not affect $(\nabla_{E_i} E_j)(p) = 0$ $\forall\ 1 \le i, j \le n$. Indeed, recall that a Gram-Schmidt consists of replacing E_i by $\frac{1}{\langle E_i, E_i \rangle^{1/2}} E_i$ or $E_i - \sum_{k \le i} \frac{\langle E_i, E_k \rangle}{\langle E_k, E_k \rangle} E_k =: E_i - f_{ik} E_k$. Let us verify the latter case (the former case is similar). By linearity, $\nabla_{E_i - f_{ik} E_k} E_j(p) = (\nabla_{E_i} E_j - f_{ik} \nabla_{E_k} E_j)(p) = 0$.
Moreover, $\nabla_{E_j}(E_i - f_{ik} E_k)(p) = \underbrace{\nabla_{E_j} E_i(p)}_{0} - f_{ik}(p) \underbrace{\nabla_{E_j} E_k(p)}_{0} - E_j(f_{ik})(p) E_k(p) = 0$

where we used $E_j \langle E_i, E_k \rangle (p) = \langle \nabla_{E_j} E_i, E_k \rangle (p) + \langle E_i, \nabla_{E_j} E_k \rangle (p) = 0$ and similarly the identity $E_j \langle E_k, E_k \rangle (p) = 0$.

8. Let M be a Riemannian manifold. Let $X \in \mathcal{X}(M)$ and $f \in \mathcal{D}(M)$. Define the divergence of X as a function $\operatorname{div} X : M \longrightarrow \mathbb{R}$ given by $\operatorname{div} X(p)$ = trace of the linear mapping $Y(p) \longrightarrow \nabla_Y X(p)$, $p \in M$, and the gradient of f as a vector field $\operatorname{grad} f$ on M defined by $\langle \operatorname{grad} f(p), v \rangle = df_p(v)$, $p \in M$, $v \in T_pM$.
a) Let E_i, $i = 1, \dots, n = \dim M$, be a geodesic frame at $p \in M$ (See Exercise 7). Show that:

$$\operatorname{grad} f(p) = \sum_{i=1}^n (E_i(f)) E_i(p),$$

$$\operatorname{div} X(p) = \sum_{i=1}^n E_i(f_i)(p), \quad \text{where } X = \sum_i f_i E_i.$$

b) Suppose that $M = \mathbb{R}^n$, with coordinates $(x_1, \dots, x_n)$ and $\frac{\partial}{\partial x_i} = (0, \dots, 1, \dots, 0) = e_i$. Show that:

$$\operatorname{grad} f = \sum_{i=1}^n \frac{\partial f}{\partial x_i} e_i$$

$$\operatorname{div} X = \sum_i \frac{\partial f_i}{\partial x_i}, \quad \text{where } X = \sum_i f_i e_i.$$

- Proof: a) • Assume $\operatorname{grad} f(p) = \sum_i a_i E_i(p)$, then $\langle \operatorname{grad} f(p), E_j(p) \rangle = a_j$, but by definition of the gradient we have $\langle \operatorname{grad} f(p), E_j(p) \rangle = df_p(E_j(p)) = E_j(f)(p)$. Thus $a_j = E_j(f)(p)$ and we have $\operatorname{grad} f(p) = \sum_{i=1}^n (E_i(f)) E_i(p)$.

• Note that $\nabla_Y X(p) = \nabla_Y\left(\sum_{i=1}^n f_i E_i\right)(p) = \sum_{i=1}^n \nabla_Y (f_i E_i)(p)$

$$= \sum_{i=1}^n \left(f_i \nabla_Y E_i(p) + Y(f_i)\big|_p E_i(p)\right) = \sum_{i=1}^n Y(f_i)\big|_p E_i(p)$$

Thus the linear mapping $Y(p) \longmapsto \nabla_Y X(p)$ has the following matrix form under the geodesic frame at p:

$$\begin{pmatrix} E_1(p) \\ \vdots \\ E_n(p) \end{pmatrix} \longmapsto \begin{pmatrix} E_1(f_1)(p) & & \\ & \ddots & \\ & & E_n(f_n)(p) \end{pmatrix} \begin{pmatrix} E_1(p) \\ \vdots \\ E_n(p) \end{pmatrix}$$

and hence $\operatorname{div} X = \sum_{i=1}^n E_i(f_i)(p)$.

b) If $M = \mathbb{R}^n$, then $E_i = \partial/\partial x_i = e_i$ is the geodesic frame at any $p \in \mathbb{R}^n$. By a) we have

$$\operatorname{grad} f(p) = \sum_{i=1}^n E_i(f) E_i(p) = \sum_{i=1}^n \frac{\partial f}{\partial x_i}\Big|_p e_i$$

$$\operatorname{div} X(p) = \sum_{i=1}^n E_i(f_i)(p) = \sum_{i=1}^n \frac{\partial f}{\partial x_i}\Big|_p$$

By the arbitrariness of $p \in \mathbb{R}^n$, we conclude that

$$\operatorname{grad} f = \sum_{i=1}^n \frac{\partial f}{\partial x_i} e_i$$

$$\operatorname{div} X = \sum_{i=1}^n \frac{\partial f_i}{\partial x_i} \quad \text{where } X = \sum_i f_i e_i.$$

9. Let M be a Riemannian manifold. Define an operator $\Delta : \mathcal{D}(M) \longrightarrow \mathcal{D}(M)$ (the Laplacian of M) by $\Delta f = \operatorname{div} \operatorname{grad} f$, $f \in \mathcal{D}(M)$.
a) Let E_i be a geodesic frame at $p \in M$, $i = 1, \cdots, n = \dim M$ (see Exercise 7). Prove that $\Delta f(p) = \sum_i E_i(E_i(f))(p)$. Conclude that if $M = \mathbb{R}^n$, Δ coincides with the usual Laplacian, namely, $\Delta f = \sum_i \frac{\partial^2 f}{\partial x_i^2}$.
b) Show that $\Delta(f \cdot g) = f\Delta g + g\Delta f + 2\langle \operatorname{grad} f, \operatorname{grad} g\rangle$.

• Proof: a) By Exercise 8, $\operatorname{div}(\operatorname{grad} f)(p) = \sum_{i=1}^n E_i(E_i(f))(p)$ since $\operatorname{grad} f = \sum_{i=1}^n (E_i(f)) E_i$.
When $M = \mathbb{R}^n$, obviously $\Delta f = \sum_{i=1}^n \partial_i \partial_i f(p) = \sum_{i=1}^n \partial^2 f/\partial x_i^2\big|_p$.
b) First note that by definition $\langle \operatorname{grad}(fg)(p), v\rangle = d(fg)_p(v) = f(p)\,dg_p(v) + g(p)\,df_p(v)$
$= f(p)\langle \operatorname{grad} g(p), v\rangle + g(p)\langle \operatorname{grad} f(p), v\rangle = \langle f(p)\operatorname{grad} g(p) + g(p)\operatorname{grad} f(p), v\rangle$,

thus $\operatorname{grad}(fg)(p) = f(p)\operatorname{grad}g(p) + g(p)\operatorname{grad}f(p)$ and thus $\operatorname{grad}(f\cdot g) = f\operatorname{grad}g + g\operatorname{grad}f$.

Secondly, note that $\operatorname{div}(f\cdot X)(p) = \sum_{i=1}^{n} E_i(f\cdot f_i)(p) = \sum_{i=1}^{n}\left(E_i(f)f_i(p) + f(p)E_i(f_i)(p)\right)$

$$= \langle \operatorname{grad}f(p), X(p)\rangle_p + f(p)\operatorname{div}X(p).$$

by the arbitrariness of $p\in M$, we have $\operatorname{div}(f\cdot X) = \langle \operatorname{grad}f, X\rangle + f\operatorname{div}X$.

Finally, we conclude that

$$\Delta(f\cdot g) = \operatorname{div}(\operatorname{grad}(f\cdot g)) = \operatorname{div}(f\cdot\operatorname{grad}g + g\cdot\operatorname{grad}f)$$

$$= \langle\operatorname{grad}f, \operatorname{grad}g\rangle + f\operatorname{div}\operatorname{grad}g + \langle\operatorname{grad}g, \operatorname{grad}f\rangle + g\operatorname{div}\operatorname{grad}f$$

$$= f\cdot\Delta g + g\cdot\Delta f + 2\langle\operatorname{grad}f, \operatorname{grad}g\rangle.$$

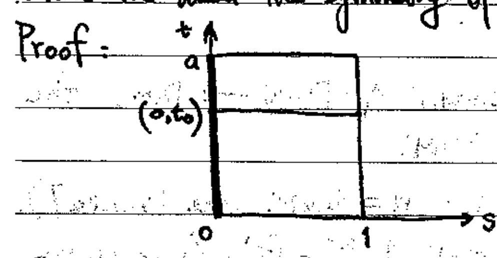
10. Let $f:[0,1]\times[0,a]\longrightarrow M$ be a parametrized surface such that for all $t_0\in[0,a]$, the curve $s\longrightarrow f(s,t_0)$, $s\in[0,1]$, is a geodesic parametrized by arc length, which is orthogonal to the curve $t\longrightarrow f(0,t)$, $t\in[0,a]$, at the point $f(0,t_0)$. Prove that, for all $(s_0,t_0)\in[0,1]\times[0,a]$, the curves $s\longrightarrow f(s,t_0)$, $t\longrightarrow f(s_0,t)$ are orthogonal.

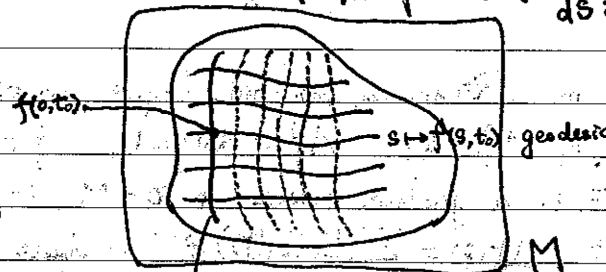
Hint: Differentiate $\left\langle\frac{\partial f}{\partial s}, \frac{\partial f}{\partial t}\right\rangle$ with respect to s, obtaining

$$\frac{d}{ds}\left\langle\frac{\partial f}{\partial s}, \frac{\partial f}{\partial t}\right\rangle = \left\langle\frac{D}{ds}\frac{\partial f}{\partial s}, \frac{\partial f}{\partial t}\right\rangle + \left\langle\frac{\partial f}{\partial s}, \frac{D}{\partial t}\frac{\partial f}{\partial s}\right\rangle = \frac{1}{2}\frac{d}{dt}\left\langle\frac{\partial f}{\partial s}, \frac{\partial f}{\partial s}\right\rangle = 0,$$

where we used the symmetry of the connection and the fact that $\frac{D}{ds}\frac{\partial f}{\partial s} = 0$.

- Proof:





$t\mapsto f(0,t)$

(沿着这族测地线以速率1平推 $t\mapsto f(0,t)$ 得到的每条线均垂直于所有测地线)

Note that $\left\langle\frac{\partial f}{\partial t}(0,t_0), \frac{\partial f}{\partial s}(0,t_0)\right\rangle_{(0,t_0)} = 0$, and that

$$\frac{d}{ds}\left\langle\frac{\partial f}{\partial t}(s_0,t_0), \frac{\partial f}{\partial s}(s_0,t_0)\right\rangle_{(s_0,t_0)} = \left\langle\frac{D}{ds}\frac{\partial f}{\partial t}(s_0,t_0), \frac{\partial f}{\partial s}(s_0,t_0)\right\rangle_{(s_0,t_0)} + \left\langle\frac{\partial f}{\partial t}(s_0,t_0), \cancel{\frac{D}{ds}\frac{\partial f}{\partial s}(s_0,t_0)}^{\,0}\right\rangle_{(s_0,t_0)}$$

$$= \left\langle\frac{D}{\partial s}\frac{\partial f}{\partial t}(s_0,t_0), \frac{\partial f}{\partial s}(s_0,t_0)\right\rangle_{(s_0,t_0)} = \left\langle\frac{D}{\partial t}\frac{\partial f}{\partial s}(s_0,t_0), \frac{\partial f}{\partial s}(s_0,t_0)\right\rangle_{(s_0,t_0)} = \frac{1}{2}\frac{d}{dt}\left\langle\frac{\partial f}{\partial s}(s_0,t_0), \frac{\partial f}{\partial s}(s_0,t_0)\right\rangle_{(s_0,t_0)}$$

$= \frac{1}{2}\frac{d}{dt}1 \equiv 0 \quad \forall (s_0,t_0)\in[0,1]\times[0,a]$. Thus $\left\langle\frac{\partial f}{\partial t}(s_0,t_0), \frac{\partial f}{\partial s}(s_0,t_0)\right\rangle_{(s_0,t_0)} \equiv \left\langle\frac{\partial f}{\partial t}(0,t_0), \frac{\partial f}{\partial s}(0,t_0)\right\rangle_{(0,t_0)} = 0.$ ∎

11. Let M be an oriented Riemannian manifold. Let ν be a differential form of degree $n = \dim M$ defined in the following way:

$$\nu(v_1, \cdots, v_n)(p) = \pm\sqrt{\det(\langle v_i, v_j\rangle)} = \text{orient. vol. } \{v_1, \cdots, v_n\}, \quad p \in M,$$

where $v_1, \cdots, v_n \in T_pM$ are linearly independent, and the oriented volume is affected by the sign $+$ or $-$ depending on whether or not the basis $\{v_1, \cdots, v_n\}$ belongs to the orientation of M; ν is called the <u>volume element</u> of M. For a vector field $X \in \mathcal{X}(M)$ define the <u>interior product</u> $i(X)\nu$ of X with ν as the $(n-1)$-form:

$$i(X)\nu(Y_2, \cdots, Y_n) = \nu(X, Y_2, \cdots, Y_n), \quad Y_2, \cdots, Y_n \in \mathcal{X}(M).$$

Prove that

$$d(i(X)\nu) = \operatorname{div} X\, \nu.$$

<u>Hint.</u> Let $p \in M$ and let E_i be a geodesic frame at p. Write X as a sum, $X = \sum f_i E_i$ and let ω_i be differential forms of degree one defined on a neighborhood of p by $\omega_i(E_j) = \delta_{ij}$. Show that $\omega_1 \wedge \cdots \wedge \omega_n$ is a volume form ν on M. Next put $\theta_i = \omega_1 \wedge \cdots \wedge \widehat{\omega_i} \wedge \cdots \wedge \omega_n$, where $\widehat{\omega_i}$ signifies that the factor ω_i is not present. Prove that $i(X)\nu = \sum_i (-1)^{i+1} f_i \theta_i$. It then follows that

$$d(i(X)\nu) = \sum_i (-1)^{i+1} df_i \wedge \theta_i + \sum_i (-1)^{i+1} f_i \wedge d\theta_i = \left(\sum_i E_i(f_i)\right)\nu + \sum_i (-1)^{i+1} f_i \wedge d\theta_i.$$

But $d\theta_i = 0$ at p, since

$$d\omega_k(E_i, E_j) = E_i\,\omega_k(E_j) - E_j\,\omega_k(E_i) - \omega_k([E_i, E_j]) = \omega_k(\nabla_{E_i}E_j - \nabla_{E_j}E_i).$$

Therefore

$$d(i(X)\nu)(p) = \left(\sum_i E_i(f_i)(p)\right)\nu = \operatorname{div} X(p)\, \nu$$

and since p is arbitrary, this completes the proof.

<u>Remark:</u> The result obtained implies that the notion of the divergence of X makes sense on an oriented differentiable manifold on which a "volume element" has been chosen, that is, an n-form ν which takes positive values on positive bases.

- Proof: Recall that a volume form is defined as an n-form on M such that $\Omega_p(E_1, \cdots, E_n) = 1$ for every $p \in M$ and every oriented orthonormal basis (E_i) for T_pM. By linear algebra, it suffices to test $\omega_1 \wedge \cdots \wedge \omega_n$ on one frame, say $\{E_i\}$

Note that by definition $\omega_i(E_j)=\delta_{ij}$ we have $(\omega_1\wedge\cdots\wedge\omega_n)(E_1,\cdots,E_n)=\omega_1(E_1)\cdots\omega_n(E_n)=1 \Rightarrow \omega_1\wedge\cdots\wedge\omega_n$ is a volume form ν on M.

Next, it is easy to show that $\{\theta_i=\omega_1\wedge\cdots\wedge\hat{\omega}_i\wedge\cdots\wedge\omega_n\}$ is a set of linear independent $(n-1)$-forms, thus they form a basis of $A^{n-1}(M)$. Since $i(X)\nu\in A^{n-1}(M)$, we may assume $i(X)\nu=\sum_{i=1}^n c_i\theta_i$. By definition we have $(i(X)\nu)(E_1,\cdots,\hat{E}_k,\cdots,E_n)=\sum_{i=1}^n c_i\theta_i(E_1,\cdots,\hat{E}_k,\cdots,E_n)$

$$= c_k\theta_k(E_1,\cdots,\hat{E}_k,\cdots,E_n)=c_k$$

On the other hand, $(i(X)\nu)(E_1,\cdots,\hat{E}_k,\cdots,E_n)=\nu(X,E_1,\cdots,\hat{E}_k,\cdots,E_n)=$
$=(-1)^{k-1}\nu(E_1,\cdots,E_{k-1},X,E_{k+1},\cdots,E_n)=(-1)^{k-1}\nu(E_1,\cdots,E_{k-1},f_kE_k,E_{k+1},\cdots,E_n)=$
$=(-1)^{k-1}f_k\nu(E_1,\cdots,E_n)=(-1)^{k-1}f_k\nu$. Thus $c_k=(-1)^{k-1}f_k=(-1)^{k+1}f_k$, and we conclude that $i(X)\nu=\sum_{k=1}^n(-1)^{k+1}f_k\theta_k$

It then follows that

$$d(i(X)\nu)=d\left(\sum_{k=1}^n(-1)^{k+1}f_k\theta_k\right)=\sum_{k=1}^n(-1)^{k+1}\left[df_k\wedge\theta_k+f_k\,d\theta_k\right]$$

$$=\sum_{k=1}^n(-1)^{k+1}df_k\wedge\theta_k+\sum_{k=1}^n(-1)^{k+1}f_k\,d\theta_k. \qquad (*)$$

We want to show $d\theta_i=0$ at p. It suffices if we prove $d\omega_k=0$ at p.

<u>Lemma</u> For any one-form ω_k and vector fields E_i,E_j, we have

$$d\omega_k(E_i,E_j)=E_i\omega_k(E_j)-E_j\omega_k(E_i)-\omega_k([E_i,E_j])$$

<u>Proof of the lemma</u>: By linearity it suffices to prove for $\omega_k=f\,dx_k$, $f\in C^\infty(M)$.

$d\omega_k=df\wedge dx_k \Rightarrow d\omega_k(E_i,E_j)=df(E_i)dx_k(E_j)-df(E_j)dx_k(E_i)=E_i(f)dx_k(E_j)-E_j(f)\,dx_k(E_i)$
$=E_i(f\,dx_k(E_j))-fE_i(dx_k(E_j))-E_j(f\,dx_k(E_i))+fE_j(dx_k(E_i))$

$$=E_i\omega_k(E_j)-E_j\omega_k(E_i)-f\left[E_i(dx_k(E_j))-E_j(dx_k(E_i))\right]$$

Assume $E_i=a_{il}\partial_l$, $E_j=a_{jm}\partial_m$, we directly verify that
$E_i(dx_k(E_j))=a_{il}\partial_l a_{jm}$, $E_j(dx_k(E_i))=a_{jl}\partial_l a_{im}$. Thus we have
$E_i(dx_k(E_j))-E_j(dx_k(E_i))=a_{il}\partial_l a_{jk}-a_{jl}\partial_l a_{ik}=dx_k([E_i,E_j])$, and that
$d\omega_k(E_i,E_j)=E_i\omega_k(E_j)-E_j\omega_k(E_i)-f[E_i(dx_k(E_j))-E_j(dx_k(E_i))]$

$$=E_i\omega_k(E_j)-E_j\omega_k(E_i)-f\,dx_k([E_i,E_j])$$

$$=E_i\omega_k(E_j)-E_j\omega_k(E_i)-\omega_k([E_i,E_j]) \quad \blacksquare$$

In our case, $\omega_k(E_j) = \delta_{kj}$, $\omega_k(E_i) = \delta_{ki}$, $[E_i, E_j] = \nabla_{E_i}E_j - \nabla_{E_j}E_i$.
By the previous lemma this gives us
$d\omega_k(E_i, E_j) = E_i\omega_k(E_j) - E_j\omega_k(E_i) - \omega_k([E_i, E_j]) = -\omega_k(\nabla_{E_i}E_j - \nabla_{E_j}E_i)$.
Since $\{E_i\}$ is a geodesic frame at p, thus $\nabla_{E_i}E_j(p) = 0$; $\nabla_{E_j}E_i(p) = 0$, and we know that $d\omega_k(E_i, E_j)(p) = -\omega_k(\nabla_{E_i}E_j - \nabla_{E_j}E_i)(p) = 0$. Hence $d\theta_k(p) = 0$ in (*) and we conclude that

$$d(i(X)\nu)(p) = \sum_{k=1}^{n}(-1)^{k+1}(df_k \wedge \theta_k)(p) + \sum_{k=1}^{n}(-1)^{k+1} f_k(p)\, d\theta_k(p)$$

$$= \sum_{k=1}^{n}(-1)^{k+1}(df_k \wedge \theta_k)(p) = \sum_{k=1}^{n}(-1)^{k+1}\sum_{i=1}^{n}E_i(f_k)(\omega_i \wedge \theta_k)(p)$$

$$= \sum_{k=1}^{n}(-1)^{k+1}(-1)^{k+1}E_k(f_k)\,\nu(p) = \left(\sum_{k=1}^{n}E_k(f_k)\right)(p)\,\nu(p)$$

$$= \operatorname{div}X(p)\,\nu(p)$$

where we used $df_k = \sum_{i=1}^{n}E_i(f_k)\,\omega_i$, which is easy to verify.
By the arbitrariness of $p \in M$, we conclude that

$$d(i(X)\nu) = (\operatorname{div}X)\,\nu.$$

12. (Theorem of Hopf). Let M be a compact orientable Riemannian manifold which is also connected. Let f be a differentiable function on M with $\Delta f \geq 0$. Then $f = \text{const}$. In particular, the harmonic functions on M, that is, those for which $\Delta f = 0$, are constant.

Hint: Take $\operatorname{grad} f = X$. Using Stokes theorem and the result of exercise 11, obtain $\int_M \Delta f\,\nu = \int_M \operatorname{div}X\,\nu = \int_M d(i(X)\nu) = \int_{\partial M} i(X)\nu = 0$.

Since $\Delta f \geq 0$, we have $\Delta f = 0$. Using again Stokes theorem on $f^2/2$, and the result of exercise 9(b), we obtain
$0 = \int_M \Delta(f^2/2)\,\nu = \int_M f\Delta f\,\nu + \int_M |\operatorname{grad} f|^2\nu = \int_M |\operatorname{grad} f|^2\nu$,
which together with the connectedness of M, implies that $f = \text{const}$.

- Proof: By Stokes theorem $\int_M \Delta f\,\nu = \int_M \operatorname{div}(\operatorname{grad} f)\,\nu = \int_M d(i(\operatorname{grad} f)\nu) = \int_{\partial M} i(\operatorname{grad} f)\,\nu = 0$ since $\partial M = \phi$. Since ν is an oriented volume form and $\Delta f \geq 0$, this implies $\Delta f = 0$. Using again Stokes theorem on $f^2/2$ and recalling $\Delta(fg) = f\Delta g + g\Delta f + 2\langle \operatorname{grad} f, \operatorname{grad} g\rangle$,

No.

Date

one sees that

$$0=\int_M \Delta(f^2/2)\nu = \frac{1}{2}\int_M (f\Delta f)\nu + \frac{1}{2}\int_M (f\Delta f)\nu + 2\cdot\frac{1}{2}\int_M \langle \operatorname{grad} f, \operatorname{grad} f\rangle \nu$$

$$=\int_M (f\Delta f)\nu + \int_M |\operatorname{grad} f|^2 \nu = \int_M |\operatorname{grad} f|^2 \nu.$$ The last equality follows from the observation $\Delta f = 0$. Again since ν is an oriented volume form and $|\operatorname{grad} f|^2 \geq 0$, we have $|\operatorname{grad} f|^2 \equiv 0$, i.e. $\operatorname{grad} f \equiv 0$. Thus f is locally constant on M. Since M is connected, we can conclude that f is constant on M.

13. Let M be a Riemannian manifold and $X \in \mathfrak{X}(M)$. Let $p \in M$ such that $X(p) \neq 0$. Choose a coordinate system $(t, x_2, \cdots, x_n)$ in a neighborhood U of p such that $\frac{\partial}{\partial t} = X$. Show that if $\nu = g\, dt \wedge dx_2 \wedge \cdots \wedge dx_n$ is a volume element of M, then $i(X)\nu = g\, dx_2 \wedge \cdots \wedge dx_n$. Conclude from this, using the result of Exercise 11, that $\operatorname{div} X = \frac{1}{g}\frac{\partial g}{\partial t}$. This proves that $\operatorname{div} X$ intuitively measures the degree of variation of the volume element of M along the trajectories of X.

- Proof: The existence of such a coordinate system has been established in our solution to Exercise 5 e). If $\nu = g\, dt \wedge dx_2 \wedge \cdots \wedge dx_n$ is a volume element of M, we claim that $i(X)\nu = g\, dx_2 \wedge \cdots \wedge dx_n$. In fact, assuming

$$i(X)\nu = h\, dx_2 \wedge \cdots \wedge dx_n + \sum_{k=2}^{n} h_k\, dt \wedge dx_2 \wedge \cdots \wedge \widehat{dx_k} \wedge \cdots \wedge dx_n$$

and noting that

$$h = (i(X)\nu)(\partial_2, \cdots, \partial_n) = \nu(\partial_t, \partial_2, \cdots, \partial_n) = g$$

$$h_k = (i(X)\nu)(\partial_t, \partial_2, \cdots, \widehat{\partial_k}, \cdots, \partial_n) = \nu(\partial_t, \partial_t, \partial_2, \cdots, \widehat{\partial_k}, \cdots, \partial_n) = 0$$

we can conclude that $i(X)\nu = g\, dx_2 \wedge \cdots \wedge dx_n$. Now use the result of Exercise 11 to see that

$$d(i(X)\nu) = d(g\, dx_2 \wedge \cdots \wedge dx_n) = \frac{\partial g}{\partial t}\, dt \wedge dx_2 \wedge \cdots \wedge dx_n$$

$$\|$$

$$\operatorname{div}(X)\nu = \operatorname{div}(X)\, g\, dt \wedge dx_2 \wedge \cdots \wedge dx_n.$$

Therefore we can conclude that $\operatorname{div}(X)\, g = \frac{\partial g}{\partial t} \Leftrightarrow \operatorname{div}(X) = \frac{1}{g}\frac{\partial g}{\partial t}$.

14. (Liouville's Theorem). Prove that if G is the geodesic field on TM then $\operatorname{div} G = 0$. Conclude from this that the geodesic flow preserves the volume of TM.

Hint: Let $p \in M$ and consider a system $(u_1, \cdots, u_n)$ of normal coordinates at p. Such coordinates are defined in a normal neighborhood $\mathcal{U}$ of p by considering an orthonormal basis $\{e_i\}$ of T_pM and taking $(u_1, \cdots, u_n)$, $q = \exp_p\left(\sum_i u_i e_i\right)$, $i = 1, \cdots, n$, as coordinates of q. In such a coordinate system, $\Gamma_{ij}^k(p) = 0$, since the geodesics that pass through p are given by linear equations. Therefore if $X = \sum x_i \frac{\partial}{\partial u_i}$, then $\operatorname{div} X(p) = \sum \frac{\partial x_i}{\partial u_i}$. Now let (u_i) be normal coordinates in a neighborhood $\mathcal{U} \subset M$ around $p \in M$ and let (u_i, v_j), $v = \sum_j v_j \frac{\partial}{\partial u_j}$, $i, j = 1, \cdots, n$, be coordinates on TM. Calculate the volume element of the natural metric of TM at (q, v), $q \in \mathcal{U}$, $v \in T_qM$, and show that it is the volume element of the product metric on $\mathcal{U} \times \mathcal{U}$ at the point (q, q) (see Exercise 2(e)). Since the divergence of G only depends on the volume element (see Exercise 11), and G is horizontal, we can calculate $\operatorname{div} G$ in the product metric. Observe that in the coordinates (u_i, v_j) we have $G(u_i) = v_i$, $G(v_j) = -\sum_{i,k} \Gamma_{ik}^j v_i v_k$, $k = 1, \cdots, n$. Since the Christoffel symbols of the product metric on $\mathcal{U} \times \mathcal{U}$ vanish at (p, p), we obtain finally, at p,

I can't see why we need to pay attention to this fact.

$$\operatorname{div} G = \sum_i \frac{\partial v_i}{\partial u_i} - \sum_j \frac{\partial}{\partial v_j}\left(\sum_{i,k} \Gamma_{ik}^j v_i v_k\right) = 0.$$

$\boxed{0^\circ.}$ We set up notations we are going to use. For details regarding the geodesic normal coordinates, see solution to Exercise 7. The expression for divergence under a geodesic frame is found in Exercise 8. Let $\{u_i\}$ be geodesic normal coordinates in a neighborhood $\mathcal{U} \subset M$ around $p \in M$. Let (u_i, v_j), $v = \sum_j v_j \frac{\partial}{\partial u_j}$, $i, j = 1, \cdots, n$ be coordinates on TM. The natural metric on TM is given in Exercise 2. Let $\left\{\frac{\partial}{\partial u_1}\Big|_{(q,\omega)}, \cdots, \frac{\partial}{\partial u_n}\Big|_{(q,\omega)}, \frac{\partial}{\partial v_1}\Big|_{(q,\omega)}, \cdots, \frac{\partial}{\partial v_n}\Big|_{(q,\omega)}\right\}$ be a basis of $T_{(q,\omega)}(TM)$.

$\boxed{1^\circ.}$ We first characterize the horizontal and vertical components of a vector in $T_{(q,\omega)}(TM)$ explicitly. By the canonical identification of T_qM with the vertical subspace of $T_{(q,\omega)}(TM)$ we introduced in the solution to Exercise 2 b), a basis for the vertical subspace of $T_{(q,\omega)}(TM)$ can be chosen as $\left\{\frac{\partial}{\partial v_1}\Big|_{(q,\omega)}, \cdots, \frac{\partial}{\partial v_n}\Big|_{(q,\omega)}\right\}$, in which each vector in the basis is understood as the covariant derivative of a vector field along a constant path at q. For a general vector $W \in T_{(q,\omega)}(TM)$, assume $W = \sum_{k=1}^n \left(Y_k \frac{\partial}{\partial u_k}\Big|_{(q,\omega)} + Z_k \frac{\partial}{\partial v_k}\Big|_{(q,\omega)}\right)$, we have that

$$\left\langle W, \frac{\partial}{\partial v_j}\Big|_{(q,\omega)} \right\rangle_{(q,\omega)} = \left\langle d\pi(W), d\pi\left(\partial/\partial v_j\big|_{(q,\omega)}\right)\right\rangle_{(q,\omega)} + \left\langle \left(Z_l + \sum_{i=1}^n \sum_{k=1}^n \Gamma_{ik}^l(q)\omega_j Y_k\right)\frac{\partial}{\partial u_l}\Big|_q, \frac{\partial}{\partial v_j}\Big|_q \right\rangle_{(q,\omega)}$$

$$= Z_j + \sum_{i=1}^n \sum_{k=1}^n \Gamma_{ik}^l(q)\omega_j Y_k.$$

No.

Date

Here we used the following observation: assume $W=\sum_{k=1}^{n}\left(Y_k\frac{\partial}{\partial u_k}\Big|_{(q,\omega)}+Z_k\frac{\partial}{\partial v_k}\Big|_{(q,\omega)}\right)\in T_{(q,\omega)}(TM)$ is generated by the curve $\alpha: I\longrightarrow TM$, $\alpha(t)=(p(t),v(t))$, where $\alpha(0)=q$, $v(0)=\omega$, $p'(0)=\sum_{k=1}^{n}Y_k\frac{\partial}{\partial u_k}\Big|_{(q,\omega)}$, $v'(0)=\sum_{k=1}^{n}Z_k\frac{\partial}{\partial v_k}\Big|_{(q,\omega)}$. Then

$$\frac{Dv}{dt}(0)=\left(\frac{dv^l}{dt}(0)+\sum_{j=1}^{n}\sum_{k=1}^{n}\Gamma_{jk}^{l}(q)\,v^j(0)\,\frac{dp^k}{dt}(0)\right)\frac{\partial}{\partial u_l}\Big|_q=\sum_{l=1}^{n}\left(Z_l+\sum_{j=1}^{n}\sum_{k=1}^{n}\Gamma_{jk}^{l}(q)\,\omega_j Y_k\right)\frac{\partial}{\partial u_l}\Big|_q.$$

Thus the component of $W=\sum_{k=1}^{n}\left(Y_k\frac{\partial}{\partial u_k}\Big|_{(q,\omega)}+Z_k\frac{\partial}{\partial v_k}\Big|_{(q,\omega)}\right)$ which lies in the vertical subspace is given by

$$W^v=\sum_{i=1}^{n}\left(Z_i+\sum_{j=1}^{n}\sum_{k=1}^{n}\Gamma_{jk}^{i}(q)\,\omega_j Y_k\right)\frac{\partial}{\partial v_i}\Big|_{(q,v)}$$ where $\frac{\partial}{\partial v_i}\Big|_q$ is already identified with $\frac{\partial}{\partial v_i}\Big|_{(q,v)}$

It follows that the component of W which lies in the horizontal subspace is

$$W^h=W-W^v=\sum_{i=1}^{n}\left(Y_i\frac{\partial}{\partial u_i}\Big|_{(q,\omega)}-\left(\sum_{j=1}^{n}\sum_{k=1}^{n}\Gamma_{jk}^{i}(q)\,\omega_j Y_k\right)\frac{\partial}{\partial v_i}\Big|_{(q,\omega)}\right).$$

In particular, for $V\in T_{(p,v)}(TM)$, since in the geodesic normal coordinates we have $\Gamma_{jk}^{i}(p)\equiv 0$ $\forall\, 1\le i,j,k\le n$, the horizontal component of V is canonically identified with the first n coordinate component of V, and the vertical component of V is canonically identified with the last n coordinate components of V. Recall from our solution to Exercise 2 e) that:

$\langle W,V\rangle_{(q,\omega)}=\langle d\pi(W),d\pi(V)\rangle_q$ if W,V are both horizontal

$\langle W,V\rangle_{(q,\omega)}=\langle W,V\rangle_q$ if W,V are both vertical

$\langle W,V\rangle_{(q,\omega)}=0$ if one of W and V is horizontal and the other vertical.

$\boxed{2^\circ.}$ Now we compute the volume form under the natural metric on TM. Let $\left\{\frac{\partial}{\partial u_1}\Big|_{(q,\omega)},\cdots,\frac{\partial}{\partial u_n}\Big|_{(q,\omega)},\frac{\partial}{\partial v_1}\Big|_{(q,\omega)},\cdots,\frac{\partial}{\partial v_n}\Big|_{(q,\omega)}\right\}$ be a basis of $T_{(q,\omega)}(TM)$. From our previous discussion, we know $\frac{\partial}{\partial u_\rho}\Big|_{(q,\omega)}$, $\frac{\partial}{\partial v_\sigma}\Big|_{(q,\omega)}$ have the decomposition

$$\frac{\partial}{\partial u_\rho}\Big|_{(q,\omega)}=\underbrace{\left[\frac{\partial}{\partial u_\rho}\Big|_{(q,\omega)}-\sum_{i=1}^{n}\sum_{j=1}^{n}\Gamma_{j\rho}^{i}(q)\,\omega_j\frac{\partial}{\partial v_i}\Big|_{(q,\omega)}\right]}_{\text{horizontal component}}+\underbrace{\sum_{i=1}^{n}\sum_{j=1}^{n}\Gamma_{j\rho}^{i}(q)\,\omega_j\frac{\partial}{\partial v_i}\Big|_{(q,\omega)}}_{\text{vertical component}}$$

$$\frac{\partial}{\partial v_\sigma}\Big|_{(q,\omega)}=\underbrace{\frac{\partial}{\partial v_\sigma}\Big|_{(q,\omega)}}_{\text{vertical component}}$$

Therefore, we can compute the Gramian matrix for the natural metric on TM directly:

$$\left\langle \frac{\partial}{\partial u_\rho}\Big|_{(q,\omega)}, \frac{\partial}{\partial u_\sigma}\Big|_{(q,\omega)} \right\rangle_{(q,\omega)} = \left\langle d\pi\left(\frac{\partial}{\partial u_\rho}\Big|_{(q,\omega)} - \sum_{i=1}^n \sum_{j=1}^n \Gamma_{j\rho}^i(q)\,\omega_j \frac{\partial}{\partial v_i}\Big|_{(q,\omega)} \right), \right.$$

$$\left. d\pi\left(\frac{\partial}{\partial u_\sigma}\Big|_{(q,\omega)} - \sum_{l=1}^n \sum_{s=1}^n \Gamma_{s\sigma}^l(q)\,\omega_s \frac{\partial}{\partial v_l}\Big|_{(q,\omega)} \right) \right\rangle_q$$

$$+ \left\langle \sum_{i=1}^n \sum_{j=1}^n \Gamma_{j\rho}^i(q)\,\omega_j \frac{\partial}{\partial v_i}\Big|_{(q,\omega)}, \sum_{l=1}^n \sum_{s=1}^n \Gamma_{s\sigma}^l(q)\,\omega_s \frac{\partial}{\partial v_l}\Big|_{(q,\omega)} \right\rangle_q$$

$$= \left\langle \frac{\partial}{\partial u_\rho}\Big|_{(q,\omega)}, \frac{\partial}{\partial u_\sigma}\Big|_{(q,\omega)} \right\rangle_q + \sum_{i=1}^n \sum_{j=1}^n \sum_{l=1}^n \sum_{s=1}^n \Gamma_{j\rho}^i(q)\,\Gamma_{s\sigma}^l(q)\,\omega_j\omega_s \left\langle \frac{\partial}{\partial v_i}\Big|_{(q,\omega)}, \frac{\partial}{\partial v_l}\Big|_{(q,\omega)} \right\rangle_q$$

$$= g_{\rho\sigma}(q) + g_{il}(q)\,\Gamma_{j\rho}^i(q)\,\Gamma_{s\sigma}^l(q)\,\omega_j\omega_s = \left[g_{\rho\sigma} + g_{il}\Gamma_{j\rho}^i \Gamma_{s\sigma}^l \omega_j \omega_s \right](q)$$

where we used Einstein's summation convention for indices i, j, l, s.

$$\left\langle \frac{\partial}{\partial u_\rho}\Big|_{(q,\omega)}, \frac{\partial}{\partial v_\sigma}\Big|_{(q,\omega)} \right\rangle_{(q,\omega)} = \left\langle \sum_{i=1}^n \sum_{j=1}^n \Gamma_{j\rho}^i(q)\,\omega_j \frac{\partial}{\partial v_i}\Big|_{(q,\omega)}, \frac{\partial}{\partial v_\sigma}\Big|_{(q,\omega)} \right\rangle_q$$

$$= \sum_{j=1}^n \sum_{i=1}^n g_{i\sigma}(q)\,\Gamma_{j\rho}^i(q)\,\omega_j = \left[g_{i\sigma}\Gamma_{j\rho}^i \omega_j \right](q)$$

$$\left\langle \frac{\partial}{\partial v_\rho}\Big|_{(q,\omega)}, \frac{\partial}{\partial v_\sigma}\Big|_{(q,\omega)} \right\rangle_{(q,\omega)} = \left\langle \frac{\partial}{\partial v_\rho}\Big|_q, \frac{\partial}{\partial v_\sigma}\Big|_q \right\rangle_q = g_{\rho\sigma}(q)$$

Dropping the dependencies on $q \in U \subset M$ for notational simplicity, the Gramian matrix looks like

$$G = \left(\begin{array}{ccc|ccc}
g_{11} + g_{li}\Gamma_{j1}^i\Gamma_{s1}^l \omega_j\omega_s & \cdots & g_{1n} + g_{li}\Gamma_{j1}^i\Gamma_{sn}^l \omega_j\omega_s & g_{1i}\Gamma_{j1}^i\omega_j & \cdots & g_{ni}\Gamma_{j1}^i\omega_j \\
\vdots & & \vdots & \vdots & & \vdots \\
g_{n1} + g_{li}\Gamma_{jn}^i\Gamma_{s1}^l \omega_j\omega_s & \cdots & g_{nn} + g_{li}\Gamma_{jn}^i\Gamma_{sn}^l \omega_j\omega_s & g_{1i}\Gamma_{jn}^i\omega_j & \cdots & g_{ni}\Gamma_{jn}^i\omega_j \\
\hline
g_{1i}\Gamma_{j1}^i\omega_j & \cdots & g_{1i}\Gamma_{jn}^i\omega_j & g_{11} & \cdots & g_{1n} \\
\vdots & & \vdots & \vdots & & \vdots \\
g_{ni}\Gamma_{j1}^i\omega_j & \cdots & g_{ni}\Gamma_{jn}^i\omega_j & g_{n1} & \cdots & g_{nn}
\end{array} \right)$$

When computing the determinant, first we fix $k \in \mathbb{N}$, $1 \le k \le n$, and for $l = 1, \cdots, n$ multiply the $(n+l)$-th row by $-\Gamma_{sk}^l \omega_s$ and add the result to the k-th row. Then let k vary from 1 to n. The result becomes

$$\det G = \det \left(\begin{array}{ccc|ccc}
g_{11} & \cdots & g_{1n} & & & \\
\vdots & & \vdots & & 0 & \\
g_{n1} & \cdots & g_{nn} & & & \\
\hline
g_{1i}\Gamma_{j1}^i\omega_j & \cdots & g_{1i}\Gamma_{jn}^i\omega_j & g_{11} & \cdots & g_{1n} \\
\vdots & & \vdots & \vdots & & \vdots \\
g_{ni}\Gamma_{j1}^i\omega_j & \cdots & g_{ni}\Gamma_{jn}^i\omega_j & g_{n1} & \cdots & g_{nn}
\end{array} \right) = \det(g_{ij}) \cdot \det(g_{ij})$$

No.

Date

and hence $dvol_{TM} = \sqrt{\det G}\, du_1\wedge\cdots\wedge du_n\wedge dv_1\wedge\cdots\wedge dv_n$

$$= \sqrt{\left(\det(g_{ij})\right)^2}\; du_1\wedge\cdots\wedge du_n\wedge dv_1\wedge\cdots\wedge dv_n$$

$$= \sqrt{\det(g_{ij})}\,\sqrt{\det(g_{ij})}\; du_1\wedge\cdots\wedge du_n\wedge dv_1\wedge\cdots\wedge dv_n$$

which is identical to the volume element of the product metric on $\mathcal{U}\times\mathcal{U}$ at (q,q). (Note that the Gramian matrix of the product metric looks like
$$\left(\begin{array}{c|c}(g_{ij}) & 0\\ \hline 0 & (g_{ij})\end{array}\right)\text{.)}$$

$\boxed{3^\circ}$ By the canonical identification of the vertical subspace with the tangent plane, we can easily view a vector field on TM as a vector field on $M\times M$:
$$Y_k\frac{\partial}{\partial u_k}\Big|_{(q,w)} + Z_j\frac{\partial}{\partial v_j}\Big|_{(q,w)} \longleftrightarrow \left(Y_k d\pi\left(\frac{\partial}{\partial u_k}\Big|_{(q,w)}\right),\ Z_j\frac{\partial}{\partial u_j}\Big|_q\right)$$
$$\| $$
$$\left(Y_k\frac{\partial}{\partial u_k}\Big|_q,\ Z_j\frac{\partial}{\partial u_j}\Big|_q\right)$$
and the natural metric on TM can also be viewed as a metric on $M\times M$.
Since this metric has the same volume element as that of the product metric at $(q,q)\in M\times M$, and the divergence of a vector field only depends on the volume element, we can compute the divergence under the product metric. (All the statements here are local, i.e. TM is supposed to be $TM|_{\mathcal{U}}$, $M\times M$ is supposed to be $\mathcal{U}\times\mathcal{U}$, etc.)

$\boxed{4^\circ}$ What we are going to prove below may be a little bit more than what we need, but it is a good and easy exercise to work with.

Lemma: Suppose we have two Riemannian manifolds (M, g_M) and (N, g_N). Then the product has a natural product metric $(M\times N, g_M+g_N)$. Let $p\in M$ and $q\in N$, and let $\mathcal{U}\subset M$ and $V\subset N$ be geodesic normal neighborhood of p, q respectively. Then $\mathcal{U}\times V\subset M\times N$ is a geodesic normal neighborhood of $(p,q)\in M\times N$.

Proof of the lemma: Since it is easy to verify that tangent spaces behave well under cartesian products (i.e. $T_{(p,q)}(M\times N)\cong T_pM\times T_qN$), it suffices if we can prove that the exponential map behaves well under cartesian products (i.e. $\exp_{(p,q)}(v,w) = (\exp_p v, \exp_q w)$). By the uniqueness of geodesics starting from (p,q)

with tangent vector (v, w) in a geodesic normal neighborhood of $(p,q) \in M \times N$, it suffices to show that $\gamma(t) = (\exp_p tv, \exp_q tw) = (\gamma_M(t), \gamma_N(t))$ is a geodesic in $M \times N$. Note that for any $X \in \mathfrak{X}(M)$, $Y \in \mathfrak{X}(N)$ we have $\nabla_X Y = 0$, $\nabla_Y X = 0$, since

$\forall Z \in \mathfrak{X}(M)$, $2g(\nabla_X Y, Z) = X\langle Y, Z\rangle + Y\langle Z, X\rangle - Z\langle X, Y\rangle - \langle X, [Y,Z]\rangle + \langle Y, [Z,X]\rangle + \langle Z, [X,Y]\rangle = 0$

$\forall W \in \mathfrak{X}(N)$, $2g(\nabla_X Y, W) = X\langle Y, W\rangle + Y\langle W, X\rangle - W\langle X, Y\rangle - \langle X, [Y,W]\rangle + \langle Y, [W,X]\rangle + \langle W, [X,Y]\rangle = 0$

and similarly for $\nabla_Y X$. Here we used the obvious identification of $X \in \mathfrak{X}(M)$, $Y \in \mathfrak{X}(N)$ with vector fields on $M \times N$. Therefore, we have

$$\nabla_{\gamma'(t)} \gamma'(t) = \nabla_{\gamma_M'(t) + \gamma_N'(t)} (\gamma_M'(t) + \gamma_N'(t)) = \nabla_{\gamma_M'(t)} \gamma_M'(t) + \nabla_{\gamma_N'(t)} \gamma_M'(t) + \nabla_{\gamma_M'(t)} \gamma_N'(t) + \nabla_{\gamma_N'(t)} \gamma_N'(t)$$

$= 0 + 0 + 0 + 0 = 0$ where we used the fact that γ_M, γ_N are geodesics in M, N respectively.

Thus we proved that γ is a geodesic in $M \times N$, which completes the proof of the lemma. ∎

$\boxed{5°}$ The proof of the previous lemma actually tells us even more than what is stated in the lemma: by showing that $\nabla_X Y = 0 = \nabla_Y X$ for all $X \in \mathfrak{X}(M)$, $Y \in \mathfrak{X}(N)$ we actually realize that if $(u_1, \cdots, u_m)$ and $(v_1, \cdots, v_n)$ are geodesic normal coordinates in M, N around p, q respectively, then $(u_1, \cdots, u_m, v_1, \cdots, v_n)$ is a system of geodesic normal coordinates in $M \times N$ around (p,q). Recall in the hint we used the fact that "if $(u_1, \cdots, u_n)$ is a system of geodesic normal coordinates at p then $\{\frac{\partial}{\partial u_1}, \cdots, \frac{\partial}{\partial u_n}\}$ is a geodesic frame at p (cf. Exercise 7)" to deduce from Exercise 8 that the divergence of $X = \sum x_i \frac{\partial}{\partial u_i}$ at p is $\operatorname{div} X(p) = \sum \frac{\partial x_i}{\partial u_i}$. Here we can use the same argument for the product metric on $M \times M$ near (p,p) to conclude that the divergence of $X = \left(\sum x_i \frac{\partial}{\partial u_i}, \sum y_j \frac{\partial}{\partial v_j}\right)$ at (p,p) is $\operatorname{div} X(p,p) = \sum_{i=1}^n \left(\frac{\partial x_i}{\partial u_i} + \frac{\partial y_i}{\partial v_i}\right)$. Observe that in the coordinates (u_i, v_j) we have

$$\begin{cases} G(u_i) = v_i \\ G(v_j) = -\sum_{i,k} \Gamma_{ik}^j v_i v_k \end{cases} \quad k = 1, \cdots, n$$

thus
$$\operatorname{div} G(p,p) = \sum_{i=1}^n \frac{\partial v_i}{\partial u_i}\bigg|_{(p,p)} - \sum_{j=1}^n \frac{\partial}{\partial v_j}\left(\sum_{i,k=1}^n \Gamma_{ik}^j(p) v_i v_k\right) = 0 - 0 = 0.$$

since $\Gamma_{ik}^j(p) = 0 \quad \forall\, 1 \le i, j, k \le n$.

$\boxed{6°}$ Now by the conclusion in Exercise 13, $\operatorname{div} X = \frac{1}{g}\frac{\partial g}{\partial t}$ where $X = \frac{\partial}{\partial t}$ under suitable change of coordinates, we have that $\frac{1}{g}\frac{\partial g}{\partial t}\Big|_{(p,v)} = \operatorname{div} G(p,v) = 0$ for arbitrary $(p,v) \in TM$, where $\frac{\partial}{\partial t} = G$ under suitable change of coordinates in TM.

Since the volume form $g\, du_1 \wedge \cdots \wedge du_n \wedge dv_1 \wedge \cdots \wedge dv_n$ is always nondegenerate under a Riemannian metric, we can conclude that

$$\mathcal{L}_G(d\mathrm{vol}) = (\mathrm{div}\, G)\, d\mathrm{vol} = \frac{1}{g}\frac{\partial g}{\partial t}\, d\mathrm{vol} = 0 \quad \text{at all } (p,v) \in TM$$

which means that the geodesic flow (the local flow of the geodesic field) preserves the volume of TM. ∎

Remark 1. (formula for divergence of a vector field in general coordinates).
For a vector field $V^m \frac{\partial}{\partial x^m}$ on (M,g), the divergence is

$$\mathrm{div}\, V = \nabla_m V^m = \frac{\partial V^m}{\partial x^m} + V^k \frac{\partial \log \sqrt{|g|}}{\partial x^k} = \frac{1}{\sqrt{|g|}} \frac{\partial (V^m \sqrt{|g|})}{\partial x^m}.$$

Therefore, even if we prefer to circumvent the switch of point of view from the Sasaki metric to the product metric (as suggested in the hint) by computing directly the divergence of the geodesic field, there is essentially no way to avoid computing the volume element (or more specifically $|g|$) as we did in part 2°. (This remark was communicated by Prof. Mark Stern)

Remark 2. There is a more general notion of "horizontal - vertical decomposition" for vector bundles which is similar to the one introduced on TM by the Sasaki metric. (By the way, one may sometimes be interested in consulting Sasaki's original 1958 paper: "On the Differential Geometry of Tangent bundles of Riemannian Manifolds", Tohoku Math. J. (2) Volume 10, Number 3 (1958), 338-354.) Assume $\pi: E \to M$ is a smooth vector bundle, and $\{\sigma_1, \cdots, \sigma_n\}$ is a local frame around $p \in M$, and $(u_1, \cdots, u_n, v_1, \cdots, v_k)$ are coordinates around $E|_U$. There is always a canonical way to identify a "vertical profile" of the tangent space of the bundle: just take tangent vectors which are "tangent to the fibre"; however, there is no immediate way to specify a "horizontal profile" of the tangent space of the bundle. Note that imposing $\{\frac{\partial}{\partial u_1}, \cdots, \frac{\partial}{\partial u_n}\}$ to be horizontal is not the best, since this notion of being horizontal is not invariant on the overlapping regions of charts on M:

$$\frac{\partial}{\partial \tilde{u}_j} = \frac{\partial u_i}{\partial \tilde{u}_j}\frac{\partial}{\partial u_i} + \frac{\partial v_k}{\partial \tilde{u}_j}\frac{\partial}{\partial v_k}, \quad \text{and in general } \frac{\partial v_k}{\partial \tilde{u}_j} \neq 0.$$

The $\{\frac{\partial}{\partial v_k}\}$ components are OK since in a vector bundle the coordinate change in the fiber are elements in $GL_n(\mathbb{R})$.

But if we replace $\frac{\partial}{\partial u_j}$ by $\left(\frac{\partial}{\partial u_j} - A^j_{ia} v_i \frac{\partial}{\partial v_a}\right)$ then it becomes well-defined on overlapping regions of charts on M. Here A^j_{ia} are connection coefficients of a specified connection on the vector bundle. This means that, although there is no generally canonical specification of a "horizontal profile" in the tangent space of a vector bundle, we can use a connection on this vector bundle to produce one "horizontal profile" for ourselves.

1. Let G be a Lie group with a bi-invariant metric $\langle\, ,\rangle$. Let $X, Y, Z \in \mathfrak{X}(G)$ be unit left invariant vector fields on G.

a) Show that $\nabla_X Y = \frac{1}{2}[X,Y]$.

Hint: Use the symmetry of the connection and the fact that $\nabla_X X = 0$ (cf. Exercise 3 of Chap. 3).

b) Conclude from a) that $R(X,Y)Z = \frac{1}{4}[[X,Y],Z]$.

c) Prove that, if X and Y are orthonormal, the sectional curvature $K(\sigma)$ of G with respect to the plane σ generated by X and Y is given by $K(\sigma) = \frac{1}{4}\|[X,Y]\|^2$. Therefore, the sectional curvature $K(\sigma)$ of a Lie group with bi-invariant metric is non-negative and is zero if and only if σ is generated by vectors X, Y which commute, that is, such that $[X,Y]=0$.

- Proof: a) We know from Exercise 3 of Chapter 3 that under a bi-invariant metric the geodesics of G that starts from e are 1-parameter subgroups of G. Thus $\nabla_X X = 0$ for all $X \in \mathfrak{X}(G)$. In particular, this gives

$$0 = \nabla_{X+Y}(X+Y) = \nabla_X X + \nabla_X Y + \nabla_Y X + \nabla_Y Y = \nabla_X Y + \nabla_Y X.$$

Furthermore, the symmetry of the connection implies $\nabla_X Y - \nabla_Y X = [X,Y]$. Combining these two equalities, we immediately obtain $\nabla_X Y = \frac{1}{2}[X,Y]$.

b)
$$\begin{aligned} R(X,Y)Z &= \nabla_Y\nabla_X Z - \nabla_X\nabla_Y Z + \nabla_{[X,Y]}Z = \nabla_Y\left(\tfrac{1}{2}[X,Z]\right) - \nabla_X\left(\tfrac{1}{2}[Y,Z]\right) + \tfrac{1}{2}[[X,Y],Z] \\ &= \tfrac{1}{4}[Y,[X,Z]] - \tfrac{1}{4}[X,[Y,Z]] + \tfrac{1}{2}[[X,Y],Z] \\ &= \tfrac{1}{4}\Big([[X,Y],Z] + [[Y,Z],X] + [[Z,X],Y]\Big) + \tfrac{1}{4}[[X,Y],Z] \\ &\underset{\text{Jacobi Identity}}{=} \tfrac{1}{4}[[X,Y],Z]. \end{aligned}$$

c) By assumption, $\langle X,X\rangle = \langle Y,Y\rangle = 1$, $\langle X,Y\rangle = 0$, and we have

$$K(\sigma) = K(X,Y) = \frac{R(X,Y,X,Y)}{\langle X,X\rangle\langle Y,Y\rangle - \langle X,Y\rangle} = R(X,Y,X,Y) = \langle R(X,Y)X, Y\rangle =$$

$$= \left\langle \tfrac{1}{4}[[X,Y],X], Y\right\rangle \underset{\uparrow}{=} -\tfrac{1}{4}\langle [X,Y],[Y,X]\rangle = \tfrac{1}{4}\langle [X,Y],[X,Y]\rangle = \tfrac{1}{4}\|[X,Y]\|^2.$$

bi-invariance of $\langle\, ,\rangle$ cf. Hint of Exercise 3, Chap. 3 or bottom formula (3) on pp.40.

For any linearly independent vectors X, Y, we can perform a Schmidt orthonormalization to obtain linearly independent vectors $\tilde{X}, \tilde{Y}$ such that $[\tilde{X},\tilde{Y}] = \lambda[X,Y]$ for some $\lambda \in \mathbb{R}$. Then $K(\sigma) = K(\tilde{X},\tilde{Y}) = \frac{1}{4}\|[\tilde X,\tilde Y]\|^2 = \frac{\lambda^2}{4}\|[X,Y]\|^2$. Therefore the sectional curvature $K(\sigma)$ of a Lie group with bi-invariant metric is non-negative and is zero if and only if σ is generated by vectors X, Y satisfying $[X,Y]=0$.

No.

Date

2. Let X be a Killing field (See Exercise 5 of Chap. 3) on a Riemannian manifold M. Define a mapping $A_X : \mathcal{X}(M) \longrightarrow \mathcal{X}(M)$ by $A_X(Z) = \nabla_Z X$, $Z \in \mathcal{X}(M)$. Consider the function $f : M \longrightarrow \mathbb{R}$ given by $f(q) = \langle X, X \rangle_q$, $q \in M$. Let $p \in M$ be a critical point of f (that is, $df_p = 0$). Prove that for any $Z \in \mathcal{X}(M)$, at p,

a) $\langle A_X(Z), X \rangle(p) = 0$.

b) $\langle A_X(Z), A_X(Z) \rangle(p) = \frac{1}{2} Z_p(Z\langle X, X \rangle) + \langle R(X,Z)X, Z \rangle(p)$.

Hint for b): Put $S = \frac{1}{2} ZZ\langle X, X \rangle - \langle R(X,Z)X, Z \rangle$. Use the Killing equation $\langle \nabla_Z X, X \rangle + \langle \nabla_X X, Z \rangle = 0$ (cf. Exercise 5 of Chap. 3), we obtain

$$S = \langle \nabla_{[X,Z]} X, Z \rangle - \langle \nabla_X X, \nabla_Z Z \rangle - \langle \nabla_X \nabla_Z X, Z \rangle.$$

Using the Killing equation again, we obtain

$$S = -\langle \nabla_Z X, \nabla_X Z \rangle + \langle \nabla_Z X, \nabla_Z X \rangle + \langle \nabla_Z X, \nabla_X Z \rangle - \langle \nabla_X X, \nabla_Z Z \rangle$$
$$= \langle \nabla_Z X, \nabla_Z X \rangle - \langle \nabla_X X, \nabla_Z Z \rangle.$$

Because of the Killing equation at p, $\nabla_X X(p) = 0$, and we conclude the assertion.

• Proof: a) By Koszul's formula, we have

$$\langle A_X(Z), X \rangle(p) = \langle \nabla_Z X, X \rangle(p) = \frac{1}{2}\{ \cancel{X\langle Z, X \rangle} + Z\langle X, X \rangle - \cancel{X\langle X, Z \rangle} - \langle \cancel{[X,X]}, Z \rangle - \cancel{\langle [Z,X], X \rangle} - \cancel{\langle [X,Z], X \rangle} \}(p)$$
$$= \frac{1}{2} Z\langle X, X \rangle(p) = \frac{1}{2} Zf(p) = 0 \quad \text{since } df_p = 0.$$

b) Let $S = \frac{1}{2} ZZ\langle X, X \rangle + \langle R(X,Z)X, Z \rangle$. By the Killing equation, we have $\langle \nabla_Z X, X \rangle + \langle \nabla_X X, Z \rangle = 0 \Rightarrow \langle \nabla_Z X, X \rangle = -\langle \nabla_X X, Z \rangle$. Thus

$$S = \frac{1}{2} ZZ\langle X, X \rangle + \langle R(X,Z)X, Z \rangle = \frac{1}{2} Z(2\langle \nabla_Z X, X \rangle) + \langle R(X,Z)X, Z \rangle$$
$$= -Z\langle \nabla_X X, Z \rangle + \langle \nabla_Z \nabla_X X - \nabla_X \nabla_Z X + \nabla_{[X,Z]} X, Z \rangle$$
$$= -\langle \nabla_Z \cancel{\nabla_X X}, Z \rangle - \langle \nabla_X X, \nabla_Z Z \rangle + \langle \nabla_Z \cancel{\nabla_X X}, Z \rangle - \langle \nabla_X \nabla_Z X, Z \rangle + \langle \nabla_{[X,Z]} X, Z \rangle$$
$$\overset{①}{=} \langle \nabla_{[X,Z]} X, Z \rangle - \langle \nabla_X X, \nabla_Z Z \rangle - \langle \nabla_X \nabla_Z X, Z \rangle$$
$$= -\langle \nabla_Z X, [X,Z] \rangle - \langle \nabla_X X, \nabla_Z Z \rangle - \langle \nabla_X \nabla_Z X, Z \rangle$$
$$= -\langle \nabla_Z X, \nabla_X Z - \nabla_Z X \rangle - \langle \nabla_X X, \nabla_Z Z \rangle - \langle \nabla_X \nabla_Z X, Z \rangle$$
$$= -\langle \nabla_Z X, \nabla_X Z \rangle + \langle \nabla_Z X, \nabla_Z X \rangle - \langle \nabla_X X, \nabla_Z Z \rangle - \langle \nabla_X \nabla_Z X, Z \rangle$$
$$\overset{②}{=} -\langle \cancel{\nabla_Z X}, \nabla_X Z \rangle + \langle \nabla_Z X, \nabla_Z X \rangle - \langle \nabla_X X, \nabla_Z Z \rangle - X\langle \cancel{\nabla_Z X, Z} \rangle^{0} + \langle \cancel{\nabla_Z X}, \nabla_X Z \rangle$$
$$= \langle \nabla_Z X, \nabla_Z X \rangle - \langle \nabla_X X, \nabla_Z Z \rangle \qquad (*)$$

where at ① we used the Killing equation $\langle \nabla_{[X,Z]} X, Z \rangle + \langle \nabla_Z X, [X,Z] \rangle = 0$; at ② we used the Killing equation $\langle \nabla_Z X, Z \rangle + \langle \nabla_Z X, Z \rangle = 0 \Rightarrow \langle \nabla_Z X, Z \rangle = 0$.

We claim that $\nabla_X X(p) = 0$. In fact, for any $Y \in \mathcal{X}(M)$, the Killing equation gives $0 = \langle \nabla_X X, Y \rangle(p) + \langle \nabla_Y X, X \rangle(p) = \langle \nabla_X X, Y \rangle(p) + \frac{1}{2} Y\langle X, X \rangle(p) = \langle \nabla_X X, Y \rangle(p) + \frac{1}{2} Yf(p)$

$= \langle \nabla_X X, Y \rangle(p)$ since $df_p = 0$. This proves the claim and we conclude from (*) that

$$\langle A_X(Z), A_X(Z) \rangle(p) = \langle \nabla_Z X, \nabla_Z X \rangle(p) = S(p) = \frac{1}{2} Z_p(Z\langle X, X \rangle) + \langle R(X,Z)X, Z \rangle(p).$$

3. Let M be a compact Riemannian manifold of even dimension whose sectional curvature is positive. Prove that every Killing field X on M has a singularity (i.e., there exists $p \in M$ such that $X(p) = 0$).

Hint: Let $f: M \longrightarrow \mathbb{R}$ be the function $f(q) = \langle X, X\rangle(q)$, $q \in M$, and let $p \in M$ be a minimum point of f (Cf. the previous Exercise). Suppose that $X(p) \neq 0$. Define a linear mapping $A: T_pM \longrightarrow T_pM$ by $A(y) = A_X Y = \nabla_Y X$, where Y is an extension of $y \in T_pM$. Let $E \subset T_pM$ be orthogonal to $X(p)$. Use the previous exercise to show that $A: E \longrightarrow E$ is an anti-symmetric isomorphism. This implies that $\dim E = \dim M - 1$ is even, which is a contradiction; thus $X(p) = 0$.

- Proof: We proceed as the hint goes. Let $f: M \longrightarrow \mathbb{R}$ be defined as $f(q) = \langle X, X\rangle(q)$, $q \in M$. Let $p = \operatorname{argmin}_{x \in M} f(x)$. If there is more than one point $x \in M$ at which f attains its minimal, just let p be any one of them. Since M is compact, p exists. Then $df_p = 0$. If $X(p) \neq 0$, we may consider the orthogonal complement of $X(p)$ in T_pM, and denote it by E. Let a linear map $A: E \longrightarrow T_pM$ be defined by $A(y) = A_X Y =$ $= \nabla_Y X$, where Y is an extension of $y \in T_pM$. Since $\nabla_Y X$ is linear in Y, $A(y)$ is well-defined in the sense that it does not depend on the extension chosen. To see that A is actually a linear mapping on E, note that from Exercise 2 a) we know $\langle A_X(Y), X\rangle(p) = 0$, thus $A_X(Y) \in E$ for indeed any $y \in T_pM$ and any extension Y. Hence $A: E \longrightarrow E$ is well-defined.
By the Killing equation, $\langle \nabla_Y X, Z\rangle + \langle \nabla_Z X, Y\rangle = 0$. Thus at point p we have $\langle A(y) + A^T(y), z\rangle = \langle \nabla_Y X, Z\rangle(p) + \langle A^T(y), z\rangle = \langle \nabla_Y X, Z\rangle(p) + \langle y, A(z)\rangle =$

$$= \langle \nabla_Y X, Z\rangle(p) + \langle Y, \nabla_Z X\rangle = 0. \quad \text{i.e. } A + A^T = 0.$$

To see that A is an isomorphism on E, note that from Exercise 2 b) we know that $\langle A(y), A(y)\rangle = \langle A_X(Y), A_X(Y)\rangle(p) = \frac{1}{2} Y_p(Y\langle X, X\rangle) + \langle R(X,Y)X, Y\rangle(p)$. Since $X(p) \neq 0$, and the sectional curvature of M is positive, and M is compact, we have $\langle R(X,Y)X, Y\rangle(p) \geq \sigma > 0$ for some positive real number σ if $Y \neq 0$. Further, since $df_p = 0$, we know $Y_p(Y\langle X, X\rangle) = 0$. It follows immediately that $\langle A(y), A(y)\rangle \geq \sigma > 0$ if $y \neq 0$. Thus A is injective. Note that A is a linear map on a finite dimensional vector space E, by the dimension theorem we know that A being injective implies that A is an isomorphism on E.

No.

Date

Now we know that $A: E \to E$ is an anti-symmetric isomorphism on E, and $\dim E = \dim M - 1$ is an odd number since $\dim M$ is an even number. Note that $\det(\lambda I - A) = \det(\lambda I + A^T) = \det(\lambda I + A) = (-1)^{\dim E}\det(-\lambda I - A) = -\det(-\lambda I - A)$. Thus if λ is an eigenvalue of A, so is $-\lambda$. But $\dim E$ is odd, thus A is an odd dimensional square matrix. Hence there is at least one eigenvalue μ of A satisfying $\mu = -\mu$, in other words A has at least one zero eigenvalue. This contradicts the fact that A is an isomorphism on E. This contradiction means that we must have $X(p) = 0$.

4. Let M be a Riemannian manifold with the following property: given any two points $p, q \in M$, the parallel transport from p to q does not depend on the curve that joins p to q. Prove that the curvature of M is identically zero, that is, for all $X, Y, Z \in \mathfrak{X}(M)$, $R(X,Y)Z = 0$

Hint: Consider a parametrized surface $f: \mathcal{U} \subset \mathbb{R}^2 \to M$, where

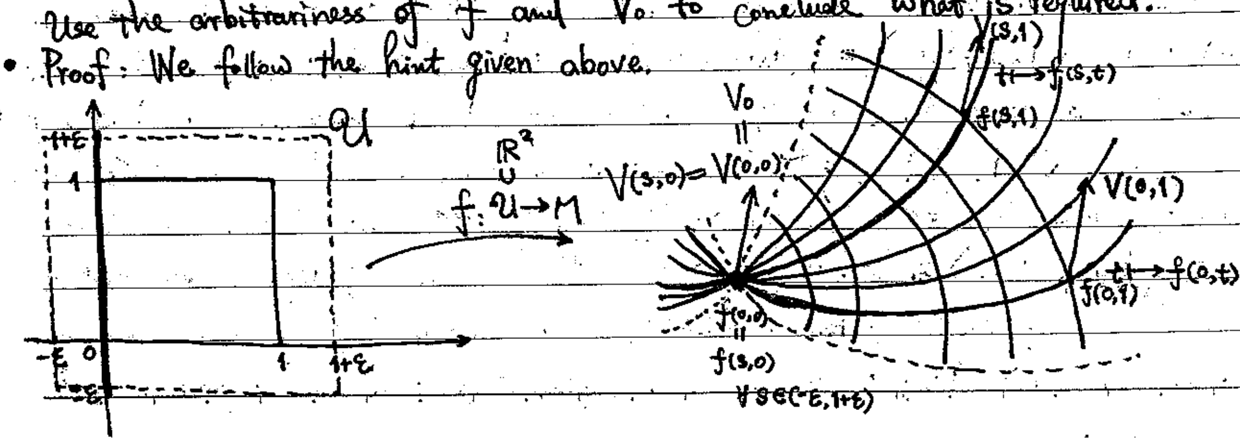
$$\mathcal{U} = \{(s,t) \in \mathbb{R}^2;\ -\varepsilon < t < 1+\varepsilon,\ -\varepsilon < s < 1+\varepsilon,\ \varepsilon > 0\}$$

and $f(s,0) = f(0,0)$, for all s. Let $V_0 \in T_{f(0,0)}(M)$ and define a field V along f by: $V(s,0) = V_0$ and, if $t \neq 0$, $V(s,t)$ is the parallel transport of V_0 along the curve $t \mapsto f(s,t)$. Then, from lemma 4.1,

$$\frac{D}{\partial s}\frac{D}{\partial t}V = 0 = \frac{D}{\partial t}\frac{D}{\partial s}V + R\left(\frac{\partial f}{\partial t}, \frac{\partial f}{\partial s}\right)V.$$

Since parallel transport does not depend on the curve chosen, $V(s,1)$ is the parallel transport of $V(0,1)$ along the curve $s \mapsto f(s,1)$, hence $\frac{D}{\partial s}V(s,1) = 0$. Thus, $R_{f(0,1)}\left(\frac{\partial f}{\partial t}(0,1), \frac{\partial f}{\partial s}(0,1)\right)V(0,1) = 0$.
Use the arbitrariness of f and V_0 to conclude what is required.

- Proof: We follow the hint given above.



Let $f: \mathcal{U} \subset \mathbb{R}^2 \longrightarrow M$ be the parametrized surface given in the hint. By Lemma 4.1 (cf. P98), we have $\frac{D}{\partial t}\frac{D}{\partial s}V - \frac{D}{\partial s}\frac{D}{\partial t}V = R\left(\frac{\partial f}{\partial s}, \frac{\partial f}{\partial t}\right)V$.
Note that for $t \neq 0$ $V(s,t)$ is the parallel transport of $V_0 = V(0,0)$ along the curve $t \mapsto f(s,t)$. Thus $\frac{D}{\partial t}V \equiv 0$ and $\frac{D}{\partial s}\frac{D}{\partial t}V = 0$. Hence $-\frac{D}{\partial t}\frac{D}{\partial s}V + R\left(\frac{\partial f}{\partial s}, \frac{\partial f}{\partial t}\right)V \equiv 0$. Moreover, by hypothesis the parallel transport does not depend on the curve chosen connecting the starting and the ending point. Hence $V(s,1)$ is also the parallel transport of $V(0,1)$ along the curve $s \mapsto f(s,1)$. Thus $\frac{D}{\partial s}V = 0$ and hence $\frac{D}{\partial t}\frac{D}{\partial s}V \equiv 0$. It follows immediately that

$$R\left(\left.\frac{\partial f}{\partial s}\right|_{\substack{s=0\\t=1}}, \left.\frac{\partial f}{\partial t}\right|_{\substack{s=0\\t=1}}\right)V(0,1) = 0 \quad \left(\text{in fact } R\left(\frac{\partial f}{\partial s}(s,t), \frac{\partial f}{\partial t}(s,t)\right)V(s,t) \equiv 0\right).$$

Since we can choose $V(0,1)$ arbitrarily (and then determine V_0, since parallel transport is "invertible"), this implies $R\left(\left.\frac{\partial f}{\partial s}\right|_{(0,1)}, \left.\frac{\partial f}{\partial t}\right|_{(0,1)}\right) = 0$. By the arbitrariness of choices of f, we can let $\left.\frac{\partial f}{\partial s}\right|_{(0,1)}$, $\left.\frac{\partial f}{\partial t}\right|_{(0,1)}$ vary through all tangent vectors in $T_{f(0,1)}M$, and let $f(0,1)$ vary through all points on M. Thus the curvature tensor $R \equiv 0$ at any point on M. In other words, the curvature of M is identically zero.

5. Let $\gamma: [0,l] \longrightarrow M$ be a geodesic and let $X \in \mathfrak{X}(M)$ be such that $X(\gamma(0)) = 0$. Show that $\nabla_{\gamma'}(R(\gamma', X)\gamma')(0) = (R(\gamma', X')\gamma')(0)$, where $X' = \frac{DX}{dt}$.

Hint: Let R be the curvature tensor of Example 5.2. Observe that, for all $Z \in \mathfrak{X}(M)$, and $t = 0$, $0 = (\nabla_{\gamma'}R)(\gamma', X, \gamma', Z) = \frac{d}{dt}\langle R(\gamma', X)\gamma', Z\rangle - \langle R(\gamma', X)\gamma', Z'\rangle - \langle R(\gamma', X)\gamma', Z'\rangle$
$= \langle \nabla_{\gamma'}(R(\gamma', X)\gamma'), Z\rangle - \langle R(\gamma', X')\gamma', Z\rangle$.

• Proof: Fix an arbitrary $z \in T_{\gamma(0)}M$, and let Z be an arbitrary extension of z along the geodesic γ. Since $X(\gamma(0)) = 0$, by the tensorial property of ∇R we get $(\nabla_{\gamma'}R)(\gamma', X, \gamma', Z)(0) = 0$. On the other hand, by definition
$(\nabla_{\gamma'}R)(\gamma', X, \gamma', Z) = (\gamma')\langle R(\gamma', X)\gamma', Z\rangle - \langle R(\overset{0}{\nabla_{\gamma'}\gamma'}, X)\gamma', Z\rangle(0) - \langle R(\gamma', \frac{DX}{dt})\gamma', Z\rangle(0)$
$- \langle R(\gamma', X)(\overset{0}{\nabla_{\gamma'}\gamma'}), Z\rangle(0) - \langle R(\gamma', X)\gamma', \frac{DZ}{dt}\rangle(0) = \left.\frac{d}{dt}\right|_{t=0}\langle R(\gamma', X)\gamma', Z\rangle \circ \gamma(t)$
$- \langle R(\gamma', X')\gamma', Z\rangle(0) - \langle R(\gamma', X)\gamma', \frac{DZ}{dt}\rangle(0) = \langle \nabla_{\gamma'}(R(\gamma', X)\gamma')(0), z\rangle_{\gamma(0)} + \cancel{\langle R(\gamma', X)\gamma', \frac{DZ}{dt}\rangle(0)}$
$- \langle R(\gamma', X')\gamma', Z\rangle(0) - \cancel{\langle R(\gamma', X)\gamma', \frac{DZ}{dt}\rangle(0)}$

Thus $\langle \nabla_{\gamma'}(R(\gamma',X)\gamma'), Z\rangle(0) = \langle R(\gamma',X')\gamma', Z\rangle(0)$. By the arbitrariness of choices of $Z \in T_{\gamma(0)}M$, this implies $\nabla_{\gamma'}(R(\gamma',X)\gamma')(0) = (R(\gamma',X')\gamma')(0)$ as desired.

6. (Locally symmetric spaces). Let M be a Riemannian manifold. M is a locally symmetric space if $\nabla R = 0$, where R is the curvature tensor of M. (The geometric significance of this condition will be given in Exercise 14 of Chap. 8).

a) Let M be a locally symmetric space and let $\gamma : [0,\ell) \to M$ be a geodesic of M. Let X, Y, Z be parallel vector fields along γ. Prove that $R(X,Y)Z$ is a parallel field along γ.

b) Prove that if M is locally symmetric, connected and has dimension two, then M has constant sectional curvature.

c) Prove that if M has constant (sectional) curvature, then M is a locally symmetric space.

- Proof: a) Choose an arbitrary vector field W along γ, we have

$$0 \equiv (\nabla R)(X,Y,Z,W,\gamma') = (\gamma')(R(X,Y,Z,W)) - \cancel{R(\nabla_{\gamma'}X,Y,Z,W)} - \cancel{R(X,\nabla_{\gamma'}Y,Z,W)}$$
$$- \cancel{R(X,Y,\nabla_{\gamma'}Z,W)} - R(X,Y,Z,\nabla_{\gamma'}W)$$
$$= \langle \nabla_{\gamma'}(R(X,Y)Z), W\rangle + \cancel{\langle R(X,Y)Z, \nabla_{\gamma'}W\rangle} - \cancel{R(X,Y,Z,\nabla_{\gamma'}W)}$$
$$= \langle \nabla_{\gamma'}(R(X,Y)Z), W\rangle.$$

By the arbitrariness of $W \in \mathfrak{X}(M)$ along γ, we conclude that $\nabla_{\gamma'}(R(X,Y)Z) \equiv 0$, i.e. $R(X,Y)Z$ is a parallel field along γ.

b) At an arbitrary point $p \in M$, let $\{E_1, E_2\}$ be a geodesic frame at p. Then $\nabla_{E_i}E_j(p) = 0 \ \forall 1 \le i,j \le 2$. Since M is locally symmetric, one has

$$E_j\langle R(E_1,E_2)E_1,E_2\rangle(p) = (\nabla R)(E_1,E_2,E_1,E_2,E_j) + R(\cancel{\nabla_{E_j}E_1}, E_2, E_1, E_2) + R(E_1, \cancel{\nabla_{E_j}E_2}, E_1, E_2)$$
$$+ R(E_1,E_2,\cancel{\nabla_{E_j}E_1},E_2) + R(E_1,E_2,E_1,\cancel{\nabla_{E_j}E_2})$$
$$= (\nabla R)(E_1,E_2,E_1,E_2,E_j) = 0 \quad \forall j = 1,2.$$

Hence $\langle R(E_1,E_2)E_1,E_2\rangle$ is locally constant in a neighborhood around $p \in M$. Since M is connected, and $\dim M = 2$, we know that M has constant sectional curvature. (We do need the condition $\dim M = 2$, for otherwise one can use the argument above to show that $K(\sigma)$ is locally constant for each 2-plane $\sigma \subset T_pM$, but has no guarantee that $K(\sigma_1) = K(\sigma_2)$ for different $\sigma_1, \sigma_2 \subset T_pM$.)

c) By Lemma 3.4 (cf. P.96), we know $R(X,Y,W,T)=K_0[\langle X,W\rangle\langle Y,T\rangle-\langle Y,W\rangle\langle X,T\rangle]$ for some constant K_0. If $K_0=0$, the conclusion trivially holds. If $K_0\neq 0$, one has

$$\frac{1}{K_0}(\nabla R)(X,Y,W,T,Z)=\frac{1}{K_0}Z\langle R(X,Y,W,T)\rangle-\frac{1}{K_0}R(\nabla_Z X,Y,W,T)-\frac{1}{K_0}R(X,\nabla_Z Y,W,T)$$

K_0 is a constant, not only a scalar function on M.

$$-\frac{1}{K_0}R(X,Y,\nabla_Z W,T)-\frac{1}{K_0}R(X,Y,W,\nabla_Z T)$$

$$=[Z\langle X,W\rangle]\langle Y,T\rangle+\langle X,W\rangle[Z\langle Y,T\rangle]-[Z\langle Y,W\rangle]\langle X,T\rangle-\langle Y,W\rangle[Z\langle X,T\rangle]$$
$$-\langle\nabla_Z X,W\rangle\langle Y,T\rangle+\langle Y,W\rangle\langle\nabla_Z X,T\rangle-\langle X,W\rangle\langle\nabla_Z Y,T\rangle+\langle\nabla_Z Y,W\rangle\langle X,T\rangle$$
$$-\langle X,\nabla_Z W\rangle\langle Y,T\rangle+\langle Y,\nabla_Z W\rangle\langle X,T\rangle-\langle X,W\rangle\langle Y,\nabla_Z T\rangle+\langle Y,W\rangle\langle X,\nabla_Z T\rangle$$
$$=[Z\langle X,W\rangle]\langle Y,T\rangle-[Z\langle X,W\rangle]\langle Y,T\rangle+\langle X,W\rangle[Z\langle Y,T\rangle]-\langle X,W\rangle[Z\langle Y,T\rangle]$$
$$-[Z\langle Y,W\rangle]\langle X,T\rangle+[Z\langle Y,W\rangle]\langle X,T\rangle-\langle Y,W\rangle[Z\langle X,T\rangle]+\langle Y,W\rangle[Z\langle X,T\rangle].$$
$$=0+0+0+0=0$$

for any $X,Y,W,T,Z\in\mathfrak{X}(M)$. Thus $\frac{1}{K_0}\nabla R\equiv 0$ and hence $\nabla R=0$, i.e. M is a locally symmetric space.

Remark: Our separation of cases $K_0=0$ and $K_0\neq 0$ is rather redundant and non-necessary... just because we want to write $1/K_0$ in front of ∇R for the purpose of simplifying formulas involving $R(X,Y,W,T)$. It turns out not to be of much help, and thus one may directly expand $(\nabla R)(X,Y,W,T,Z)$ with the harmless multiplicative constant K_0 in front of all the anoying formulae remained in the proof above. Also, note that we do need constant sectional curvature assumption here: if M is only isotropic, there is generally no guarantee that K_0 is a constant (only a scalar function on M).

7. Prove the 2nd Bianchi Identity:

$$\nabla R(X,Y,Z,W,T)+\nabla R(X,Y,W,T,Z)+\nabla R(X,Y,T,Z,W)=0$$

for all $X,Y,Z,W,T\in\mathfrak{X}(M)$.

Hint: Since the objects involved are all tensors, it suffices to prove the equality at a point $p\in M$. Choose a geodesic frame $\{e_i\}$ based at p (See Exercise 7 of Chap. 3). In this frame $\nabla_{e_i}e_j(p)=0$, hence

$$\nabla R(e_i,e_j,e_k,e_l,e_h)=e_h\langle R(e_i,e_j)e_k,e_l\rangle=e_h\langle R(e_k,e_l)e_i,e_j\rangle$$
$$=\langle\nabla_{e_h}\nabla_{e_l}\nabla_{e_k}e_i-\nabla_{e_h}\nabla_{e_k}\nabla_{e_l}e_i+\nabla_{e_h}\nabla_{[e_k,e_l]}e_i,e_j\rangle.$$

Therefore, using the Jacobi Identity for the bracket, we find

$$\nabla R(e_i,e_j,e_k,e_l,e_h)+\nabla R(e_i,e_j,e_l,e_h,e_k)+\nabla R(e_i,e_j,e_h,e_k,e_l)=$$
$$=R(e_l,e_k,\nabla_{e_h}e_i,e_j)+R(e_h,e_k,\nabla_{e_l}e_i,e_j)+R(e_h,e_l,\nabla_{e_k}e_i,e_j)=0,$$

since each one of the summands vanishes at p. The general case follows by linearity.

- Proof: We follow the hint closely. Let $\{e_i\}$ be a geodesic frame at p. $\nabla_{e_i}e_j(p)=0$.

No.

Date

Note that $\nabla R(e_i, e_j, e_k, e_l, e_h)(p) = \langle \nabla_{e_h}\nabla_{e_l}\nabla_{e_k} e_i - \nabla_{e_h}\nabla_{e_k}\nabla_{e_l} e_i + \nabla_{e_h}\nabla_{[e_k,e_l]} e_i, e_j \rangle (p)$

$\nabla R(e_i, e_j, e_l, e_h, e_k)(p) = \langle \nabla_{e_k}\nabla_{e_h}\nabla_{e_l} e_i - \nabla_{e_k}\nabla_{e_l}\nabla_{e_h} e_i + \nabla_{e_k}\nabla_{[e_l,e_h]} e_i, e_j \rangle (p)$

$\nabla R(e_i, e_j, e_h, e_k, e_l)(p) = \langle \nabla_{e_l}\nabla_{e_k}\nabla_{e_h} e_i - \nabla_{e_l}\nabla_{e_h}\nabla_{e_k} e_i + \nabla_{e_l}\nabla_{[e_h,e_k]} e_i, e_j \rangle (p)$

Thus $\nabla R(e_i, e_j, e_k, e_l, e_h) + \nabla R(e_i, e_j, e_l, e_h, e_k) + \nabla R(e_i, e_j, e_h, e_k, e_l) =$

$= \langle R(e_l, e_h)\nabla_{e_k} e_i - \nabla_{[e_l,e_h]}\nabla_{e_k} e_i + R(e_h, e_k)\nabla_{e_l} e_i - \nabla_{[e_h,e_k]}\nabla_{e_l} e_i + R(e_k, e_l)\nabla_{e_h} e_i - \nabla_{[e_k,e_l]}\nabla_{e_h} e_i$

$+ \nabla_{e_h}\nabla_{[e_k,e_l]} e_i + \nabla_{e_k}\nabla_{[e_l,e_h]} e_i + \nabla_{e_l}\nabla_{[e_h,e_k]} e_i, e_j \rangle (p) =$

$= \langle R(e_l, e_h)\nabla_{e_k} e_i + R(e_h, e_k)\nabla_{e_l} e_i + R(e_k, e_l)\nabla_{e_h} e_i$

$+ R([e_k, e_l], e_h) e_i - \nabla_{[[e_k,e_l],e_h]} e_i + R([e_l, e_h], e_k) e_i - \nabla_{[[e_l,e_h],e_k]} e_i$

$+ R([e_h, e_k], e_l) e_i - \nabla_{[[e_h,e_k],e_l]} e_i, \quad e_j \rangle (p)$

$= R(e_l, e_h, \nabla_{e_k} e_i, e_j)(p) + R(e_h, e_k, \nabla_{e_l} e_i, e_j)(p) + R(e_k, e_l, \nabla_{e_h} e_i, e_j)(p)$

$+ R([e_k, e_l], e_h, e_i, e_j)(p) + R([e_l, e_h], e_k, e_i, e_j)(p) + R([e_h, e_k], e_l, e_i, e_j)(p)$

$- \langle \nabla_{([[e_k,e_l],e_h] + [[e_l,e_h],e_k] + [[e_h,e_k],e_l])} e_i, e_j \rangle (p)$

Since $\nabla_{e_i} e_j (p) = 0$ and R is a tensor, the first six terms all vanish at p (note that $[e_k, e_l](p) = \nabla_{e_k} e_l(p) - \nabla_{e_l} e_k(p) = 0 - 0 = 0$, etc.). The last term vanishes identically because of the Jacobi Identity:

$[[e_k, e_l], e_h] + [[e_l, e_h], e_k] + [[e_h, e_k], e_l] = 0$ identically in the chosen neighborhood of p where the geodesic frame at p is defined.

By linearity, for any $X, Y, Z, W, T \in \mathfrak{X}(M)$ we have

$\nabla R(X, Y, Z, W, T)(p) + \nabla R(X, Y, W, T, Z)(p) + \nabla R(X, Y, T, Z, W)(p) = 0.$

Since $\nabla R : \mathfrak{X}(M) \times \mathfrak{X}(M) \times \mathfrak{X}(M) \times \mathfrak{X}(M) \times \mathfrak{X}(M) \longrightarrow \mathcal{D}(M)$ is a tensor, the identity above is preserved under change of coordinates (the invariance under change of frames follows from linearity though). Finally by the arbitrariness of $p \in M$, we conclude the 2nd Bianchi Identity

$\nabla R(X, Y, Z, W, T) + \nabla R(X, Y, W, T, Z) + \nabla R(X, Y, T, Z, W) = 0$ for all $X, Y, Z, W, T \in \mathfrak{X}(M)$.

Remark: The tensorial property of ∇R plays a key role here. For comparison, consider the Christoffel symbol Γ, which is not a tensor. In geodesic normal coordinates, one can let $\Gamma_{ij}^k(p) = 0$ for all i, j, k, but generally can not make Γ identically vanish in a neighborhood of p. This is an immediate consequence of the lack of tensoriality under change of coordinates.

8. (Schur's Theorem). Let M^n be a connected Riemannian manifold with $n \geq 3$. Suppose that M is isotropic, that is, for each $p \in M$, the sectional curvature $K(p,\sigma)$ does not depend on $\sigma \subset T_pM$. Prove that M has constant sectional curvature, that is, $K(p,\sigma)$ also does not depend on p.

Hint: Define a tensor R' of order 4 by

$$R'(W,Z,X,Y) = \langle W,X\rangle\langle Z,Y\rangle - \langle Z,X\rangle\langle W,Y\rangle.$$

If $K(p,\sigma) = K$ does not depend on σ, by Lemma 3.4, $R = KR'$. Therefore, for all $U \in \mathcal{X}(M)$, $\nabla_U R = (UK)R'$. Using the 2nd Bianchi identity (see Exercise 7): $\nabla R(W,Z,X,Y,U) + \nabla R(W,Z,Y,U,X) + \nabla R(W,Z,U,X,Y) = 0$, we obtain, for all $X,Y,W,Z,U \in \mathcal{X}(M)$, $0 = (UK)(\langle W,X\rangle\langle Z,Y\rangle - \langle Z,X\rangle\langle W,Y\rangle) + (XK)(\langle W,Y\rangle\langle Z,U\rangle - \langle Z,Y\rangle\langle W,U\rangle) + (YK)(\langle W,U\rangle\langle Z,X\rangle - \langle Z,U\rangle\langle W,X\rangle)$. Fix $p \in M$. Because $n \geq 3$, it is possible, fixing X at p, to choose Y and Z at p such that $\langle X,Y\rangle = \langle Y,Z\rangle = \langle Z,X\rangle = 0$, $\langle Z,Z\rangle = 1$. Put $U = Z$ at p. The relation above yields, for all W, $\langle (XK)Y - (YK)X, W\rangle = 0$. Since X and Y are linearly independent at p, we conclude that $XK = 0$ for all $X \in T_pM$. Thus $K = $ const.

- Proof: We follow the hint closely. For any $p \in M$, define $k(p) = K(p,\sigma)$. By hypothesis $k : M \to \mathbb{R}$ is well-defined. Also k is obviously smooth. By Lemma 3.4 (note that Lemma 3.4 is stated for vectors in T_pM, not for vector fields on M) we have $R(X,Y,Z,W) = k R'(X,Y,Z,W)$ for any $X,Y,Z,W \in \mathcal{X}(M)$ (not just for vectors in T_pM, but vector fields on M). For any smooth vector field U we have

$$\begin{aligned}(\nabla_U R)(X,Y,Z,W) &= U(R(X,Y,Z,W)) - R(\nabla_U X, Y, Z, W) - R(X, \nabla_U Y, Z, W) \\ &\quad - R(X,Y,\nabla_U Z, W) - R(X,Y,Z,\nabla_U W) \\ &= (Uk)\cdot R'(X,Y,Z,W) + k\cdot U(R'(X,Y,Z,W)) - k\cdot R'(\nabla_U X, Y, Z, W) \\ &\quad - k\cdot R'(X,\nabla_U Y, Z, W) - k\cdot R'(X,Y,\nabla_U Z, W) - k\cdot R'(X,Y,Z,\nabla_U W) \\ &= (Uk)(\langle X,Z\rangle\langle Y,W\rangle - \langle X,W\rangle\langle Y,Z\rangle)\end{aligned}$$

where in the last equality we used the identity

$$U(R'(X,Y,Z,W)) = R'(\nabla_U X, Y, Z, W) + R'(X,\nabla_U Y, Z, W) + R'(X,Y,\nabla_U Z, W) + R'(X,Y,Z,\nabla_U W)$$

which we have already derived once in the solution for Exercise 6 c) in this chapter. By the 2nd Bianchi Identity, one obtain

$$\begin{aligned}0 &= (\nabla_U R)(X,Y,Z,W) + (\nabla_W R)(X,Y,U,Z) + (\nabla_Z R)(X,Y,W,U) \\ &= (Uk)(\langle X,Z\rangle\langle Y,W\rangle - \langle X,W\rangle\langle Y,Z\rangle) + (Wk)(\langle X,U\rangle\langle Y,Z\rangle - \langle X,Z\rangle\langle Y,U\rangle) \\ &\quad + (Zk)(\langle X,W\rangle\langle Y,U\rangle - \langle X,U\rangle\langle Y,W\rangle).\end{aligned}$$

No.

Date

Since $n \geq 3$, we can first fix W at p, $W(p) \neq 0$, and choose Z and Y at p such that $\langle W, Z\rangle(p) = 0$, $\langle Z, Y\rangle(p) = 0$, $\langle Y, W\rangle(p) = 0$, $\langle Y, Y\rangle(p) = 1$, $\langle Z, Z\rangle(p) = 1$.
Let $U = Y$ at p. The formula above implies that

$$\begin{aligned} 0 &= -(Wk)(p)\,\langle X(p), Z(p)\rangle_p \langle Y, Y\rangle(p) + (Zk)(p)\,\langle X(p), W(p)\rangle_p \langle Y, Y\rangle(p) \\ &= -(Wk)(p)\,\langle X(p), Z(p)\rangle_p + (Zk)(p)\,\langle X(p), W(p)\rangle_p \\ &= \langle X(p),\ -(Wk)(p)\,Z(p) + (Zk)(p)\,W(p)\rangle_p \quad \text{for any } X(p) \in T_pM. \end{aligned}$$

By our choices of Z, Y and W, $W(p)$ and $Z(p)$ are linearly independent (easy to verify). Thus $-(Wk)(p)Z(p) + (Zk)(p)W(p) = 0 \Rightarrow (Wk)(p) = 0$, $(Zk)(p) = 0$.
By the arbitrary choices of $W \in \mathfrak{X}(M)$, we have $k = \text{const}$ in a neighborhood of p. Since further we know M^n is connected, it is immediate to conclude that k is constant on the entire manifold M. Hence M has constant sectional curvature.

9. Prove that the scalar curvature $K(p)$ at $p \in M$ is given by
$$K(p) = \frac{1}{\omega_{n-1}} \int_{S^{n-1}} Ric_p(x)\, dS^{n-1}$$
where ω_{n-1} is the area of the sphere S^{n-1} in T_pM and dS^{n-1} is the area element on S^{n-1}.

Hint: Use the following general argument on quadratic forms. Consider an orthonormal basis $e_1, \dots, e_n$ in T_pM such that if $x = \sum_{i=1}^{n} x_i e_i$,
$$Ric_p(x) = \sum \lambda_i x_i^2, \quad \lambda_i \text{ real}.$$
Because $|x| = 1$, the vector $(x_1, \dots, x_n) = \nu$ is a unit normal vector on S^{n-1}. Denoting $V = (\lambda_1 x_1, \dots, \lambda_n x_n)$, and using Stokes Theorem, we obtain
$$\frac{1}{\omega_{n-1}} \int_{S^{n-1}} \left(\sum \lambda_i x_i^2\right) dS^{n-1} = \frac{1}{\omega_{n-1}} \int_{S^{n-1}} \langle V, \nu\rangle\, dS^{n-1} = \frac{1}{\omega_{n-1}} \int_{B^n} \operatorname{div} V\, dB^n,$$
where B^n is the unit ball whose boundary is $S^{n-1} = \partial B^n$. Noting that $(\mathrm{vol}(B^n))/\omega_n = 1/n$, we conclude that
$$\frac{1}{\omega_{n-1}} \int_{S^{n-1}} Ric_p(x)\, dS^{n-1} = \frac{1}{n} \operatorname{div} V = \frac{\sum \lambda_i}{n} = \frac{\sum Ric_p(e_i)}{n} = K(p).$$

- Proof: We follow the hint closely. Recall the symmetric bilinear quadratic form defined on pp. 97: $Q(x, y) =$ trace of the mapping $z \mapsto R(x, z)y$. By the spectral theorem of symmetric bilinear quadratic forms, we can choose an orthonormal basis $e_1, \cdots, e_n$ in T_pM such that for any $x = \sum_{i=1}^n x_i e_i$ one has $Ric_p(x) = \frac{1}{n-1} Q(x,x) = \sum_{i=1}^n \lambda_i x_i^2$ for λ_i real, $1 \le i \le n$. For each $x \in S^{n-1} \subset T_pM$, $|x| = 1$, and hence the vector $(x_1, \cdots, x_n) = \nu$ is a unit normal vector on S^{n-1}. Denoting $V = (\lambda_1 x_1, \cdots, \lambda_n x_n)$ as a vector field on T_pM, then by Stokes Theorem one has

$$\frac{1}{\omega_{n-1}} \int_{S^{n-1}} Ric_p(x)\, dS^{n-1} = \frac{1}{\omega_{n-1}} \int_{S^{n-1}} \left(\sum_{i=1}^n \lambda_i x_i^2 \right) dS^{n-1} = \frac{1}{\omega_{n-1}} \int_{S^{n-1}} \langle V, \nu \rangle \, dS^{n-1}$$

$$= \frac{1}{\omega_{n-1}} \int_{B_n} \operatorname{div} V \, dB^n = \frac{1}{\omega_{n-1}} \left(\sum_{i=1}^n \lambda_i \right) \operatorname{vol}(B^n)$$

$$= \frac{1}{\omega_{n-1}} \cdot \frac{\omega_{n-1}}{n} \cdot \sum_{i=1}^n \lambda_i = \left(\sum_{i=1}^n \lambda_i \right) \Big/ n = \frac{\sum_{i=1}^n Ric_p(e_i)}{n} = K(p)$$

where B^n is the unit ball in $T_pM \cong \mathbb{R}^n$, $\partial B^n = S^{n-1}$.

10. (Einstein manifolds). A Riemannian manifold M^n is called an Einstein manifold if, for all $X, Y \in \mathfrak{X}(M)$, $Ric(X,Y) = \lambda \langle X, Y \rangle$, where $\lambda : M \to \mathbb{R}$ is a real valued function. Prove that:

a) If M^n is connected and Einstein, with $n \ge 3$, then λ is constant on M.

b) If M^3 is a connected Einstein manifold then M^3 has constant sectional curvature.

Hint for (a): Consider a geodesic orthonormal frame $\{e_i\}$, $i = 1, \cdots, n \ge 3$, at a point $p \in M$ (see Exercise 7 of Chap. 3). The 2nd Bianchi identity (see Exercise 7) at p can be written

$$(*) \quad e_s(R_{hijk}) + e_j(R_{hiks}) + e_k(R_{hisj}) = 0$$

where R_{hijk} are the components of the curvature tensor in this frame and we take into account that $\nabla_{e_i} e_j(p) = 0$. Observe that $\langle e_i, e_k \rangle = g_{ik} = \delta_{ik} = \delta^{ik}$. Multiplying $(*)$ by $\delta_{ik}\delta_{hj}$ and summing on i, k, h, j, we obtain: for the first part,

$$\sum_{ikjh} \delta_{hj} \delta_{ik} e_s(R_{hijk}) = e_s\left(\sum_{ikjh} \delta_{hj} \delta_{ik} R_{hijk} \right) = e_s\left(\sum_{hj} \delta_{hj} R_{hj} \right)$$

$$= e_s\left(\sum_{hj} \delta_{hj} (\lambda \delta_{hj}) \right) = n\, e_s(\lambda);$$

for the second part,

$$\sum_{ikjh} \delta_{hj} \delta_{ik} e_j(R_{hiks}) = -\sum_{jh} \delta_{hj} e_j \left(\sum_{ik} \delta_{ik} R_{hisk} \right)$$

$$= -\sum_{jh} \delta_{hj} e_j (\lambda \delta_{hs}) = -e_s(\lambda);$$

and for the third part, $\sum_{i,k,j,h} \delta_{hj}\delta_{ik} e_k\left(R_{hiks}\right) = -e_s(\lambda)$.

Therefore, (*) implies that, for all s, $(n-2)\,e_s(\lambda)=0$. From the arbitrariness of p, λ is constant on M.

- Proof: a) Fix an arbitrary point $p\in M$. Let $\{e_i\}$ be a geodesic orthonormal frame at p, i.e. $\langle e_i, e_j\rangle(p)=\delta_{ij}$, $\nabla_{e_i}e_j(p)=0$, $\forall\, 1\le i,j\le n$. The Second Bianchi Identity gives

$$\nabla R(e_h,e_i,e_j,e_k,e_s)+\nabla R(e_h,e_i,e_k,e_s,e_j)+\nabla R(e_h,e_i,e_s,e_j,e_k)=0$$

At p, by definition $\nabla R(e_h,e_i,e_j,e_k,e_s)(p) = e_s(R(e_h,e_i,e_j,e_k))(p)-R(\nabla_{e_s}e_h,e_i,e_j,e_k)(p)$
$-R(e_h,\nabla_{e_s}e_i,e_j,e_k)(p)-R(e_h,e_i,\nabla_{e_s}e_j,e_k)(p)-R(e_h,e_i,e_j,\nabla_{e_s}e_k)(p) =$
$= e_s(R(e_h,e_i,e_j,e_k))(p)$ Since R is a tensor and $\nabla_{e_i}e_j(p)=0$. Thus

(*) $e_s(R_{hijk})(p)+e_j(R_{hiks})(p)+e_k(R_{hisj})(p)=0$ by the 2nd Bianchi Identity at p.

Observe that $\langle e_i, e_k\rangle(p)=g_{ik}(p)=\delta_{ik}=\delta^{ik}$. Thus multiplying (*) by $\delta_{ik}\delta_{hj}$ and summing on i,k,h,j gives

$$\sum_{i,k,h,j}\delta_{ik}\delta_{hj}\, e_s(R_{hijk})(p) = e_s\Big(\sum_{i,k,h,j}\delta_{ik}\delta_{hj}R_{hijk}\Big)(p) = e_s\Big(\sum_{h,j}\delta_{hj}\sum_{i,k} g^{ik}R_{hijk}\Big)(p)$$

$$= e_s\Big(\sum_{h,j}\delta_{hj}\sum_i R^{\,i}_{hij}\Big)(p) = e_s\Big(\sum_{h,j}\delta_{hj}R_{hj}\Big)(p) = e_s\Big(\sum_{h,j}\delta_{hj}\,\lambda\, g_{hj}\Big)(p) = e_s\Big(\lambda\sum_{h,j}\delta_{hj}g^{hj}\Big)(p)$$

$$= e_s\Big(\lambda\sum_{h,j}\delta_{hj}\delta_{hj}\Big)(p) = e_s(n\cdot\lambda)(p) = n\, e_s(\lambda)(p).$$

$$\sum_{i,k,h,j}\delta_{ik}\delta_{hj}\, e_j(R_{hiks})(p) = \sum_j e_j\Big(\sum_{i,k,h}\delta_{ik}\delta_{hj}R_{hiks}\Big)(p) = \sum_j e_j\Big(\sum_{i,k,h}\delta_{ik}\delta_{hj}R_{kshi}\Big)(p)$$

$$= -\sum_{j,h}\delta_{hj}\, e_j\Big(\sum_{i,k} g^{ik}R_{skhi}\Big)(p) = -\sum_{j,h}\delta_{hj}\, e_j\Big(\sum_k R^{\,k}_{skh}\Big)(p) = -\sum_{j,h}\delta_{hj}\, e_j(R_{sh})(p)$$

$$= -\sum_{j,h}\delta_{hj}\, e_j(\lambda g_{sh})(p) = -\sum_{j,h}\delta_{hj}\, e_j(\lambda\cdot\delta_{sh})(p) = -e_s(\lambda)(p).$$

$$\sum_{i,k,h,j}\delta_{ik}\delta_{hj}\, e_k(R_{hisj})(p) = -\sum_{i,k}\delta_{ik}\, e_k\Big(\sum_{h,j}\delta_{hj}R_{ihsj}\Big)(p) = -\sum_{i,k}\delta_{ik}\, e_k\Big(\sum_{h,j} g^{hj}R_{ihsj}\Big)(p)$$

$$= -\sum_{i,k}\delta_{ik}\, e_k\Big(\sum_h R^{\,h}_{ihs}\Big)(p) = -\sum_{i,k}\delta_{ik}\, e_k(R_{is})(p) = -\sum_{i,k}\delta_{ik}\, e_k(\lambda\,\delta_{is})(p) = -e_s(\lambda)(p).$$

Therefore, (*) becomes $n\, e_s(\lambda)(p) - e_s(\lambda)(p) - e_s(\lambda)(p) = 0 \Rightarrow (n-2)\,e_s(\lambda)(p)=0$.

Since $n\ge 3$ by hypothesis, this gives $e_s(\lambda)(p)=0$ for all $1\le s\le n$. Thus λ is locally constant in a neighborhood of p. By the connectedness of M^n, and the arbitrariness of $p\in M$ we conclude that $\lambda: M \longrightarrow \mathbb{R}$ is constant on M.

b) By a) we know λ is constant. Let $\{e_1, e_2, e_3\}$ be a geodesic frame at p. Then $Ric(X,Y) = \lambda\langle X,Y\rangle$ implies $\sum_{i=1}^{3} R(X, e_i, Y, e_i) = 2\lambda\langle X,Y\rangle$. Letting $X=Y=e_j$ for $j=1,2,3$ we obtain

$$\begin{cases} R_{1212} + R_{1313} = 2\lambda g_{11} \\ R_{2121} + R_{2323} = 2\lambda g_{22} \\ R_{3131} + R_{3232} = 2\lambda g_{33} \end{cases}$$

Evaluating all three equations at p and use properties of the geodesic frame, one obtain
$$\begin{cases} R_{1212}(p) + R_{1313}(p) = 2\lambda \\ R_{1212}(p) + R_{2323}(p) = 2\lambda \\ R_{1313}(p) + R_{2323}(p) = 2\lambda \end{cases}$$
which gives $R_{1212}(p) = R_{1313}(p) = R_{2323}(p) = \lambda$.

Note that for any 2-dimensional subspace σ of T_pM^3, we can always choose $\{e_1, e_2\}$ to be an orthonormal basis of σ, and find e_3 such that $\{e_1, e_2, e_3\}$ is an orthonormal basis of T_pM^3, thus by the argument above we conclude that $K(\sigma) = R_{1212}(p) = \lambda = \text{const}$. By the arbitrariness of $p \in M^3$ and $\sigma \subset T_pM$, this implies that M^3 has constant sectional curvature.

No.

Date

1. Let M be a Riemannian manifold with sectional curvature identically zero. Show that, for every $p\in M$, the mapping $\exp_p : B_\varepsilon(0)\subset T_pM \longrightarrow B_\varepsilon(p)$ is an isometry, where $B_\varepsilon(p)$ is a normal ball at p.

- Proof 1: We need to show that for any $w_1, w_2 \in T_v(T_pM)$
$$\langle (d\exp_p)_v w_1, (d\exp_p)_v w_2\rangle_{\exp_p v} = \langle w_1, w_2\rangle_v = \langle (d\exp_p)_0 w_1, (d\exp_p)_0 w_2\rangle_{\exp_p 0} = \langle (d\exp_p)_0 w_1, (d\exp_p)_0 w_2\rangle_p .$$
By the Gauss Lemma, it suffices to show this for w_1, w_2 orthogonal to v. Further, it suffices if we show $\langle (d\exp_p)_{tv} w_1, (d\exp_p)_{tv} w_2\rangle$ is independent of t, which is equivalent to showing that $\langle (d\exp_p)_{tv} tw_1, (d\exp_p)_{tv} tw_2\rangle = \text{const}\cdot t^2$ for all t. Let $J_1(t) := (d\exp_p)_{tv} tw_1$, $J_2(t) := (d\exp_p)_{tv} tw_2$, then J_1, J_2 are Jacobi fields with $J_1(0) = J_2(0) = 0$, $J_1'(0) \perp \gamma'(0)$, $J_2'(0)\perp \gamma'(0)$, where we let $\gamma(t) := \exp_p tv$.
Write $f(t) := \langle J_1(t), J_2(t)\rangle_{\exp_p tv}$, then
$$f(0) = \langle J_1(0), J_2(0)\rangle_p = 0$$
$$\frac{df}{dt}(0) = \left\langle \frac{DJ_1}{dt}(0), J_2(0)\right\rangle_p + \left\langle J_1(0), \frac{DJ_2}{dt}(0)\right\rangle_p = 0$$
$$\frac{d^2f}{dt^2}(0) = \left\langle \frac{D^2J_1}{dt^2}(0), J_2(0)\right\rangle_p + 2\left\langle \frac{DJ_1}{dt}(0), \frac{DJ_2}{dt}(0)\right\rangle_p + \left\langle J_1(0), \frac{DJ_2}{dt}(0)\right\rangle$$
$$= 2\left\langle \frac{DJ_1}{dt}(0), \frac{DJ_2}{dt}(0)\right\rangle_p = 2\langle w_1, w_2\rangle_p$$
Note that the sectional curvature of M is identically zero, and v, w_1 are linearly independent, thus
$$\frac{D^2}{dt^2}J_1 = -R(\gamma', J_1)\gamma' \Rightarrow \left\langle \frac{D^2}{dt^2}J_1, T\right\rangle(t) = 0 \quad \text{for all } t \text{ and all vector fields } T$$
along $\gamma(t)$. Indeed, by Lemma 3.4 of Chapter 4 we have
$$\langle R(\gamma', J_1)\gamma', T\rangle = K\{\langle \gamma',\gamma'\rangle\langle J_1, T\rangle - \langle \gamma', T\rangle\langle J_1, \gamma'\rangle\} = 0\cdot\langle J_1, T\rangle \equiv 0 \quad \text{since } K\equiv 0.$$
By the arbitrariness of T we get $\frac{D^2J_1}{dt^2}\equiv 0$ for all t. Similarly $\frac{D^2J_2}{dt^2}\equiv 0$.
Hence by the Leibniz rule of derivatives we know $\frac{d^kf}{dt^k}(0)\equiv 0$ for all $k\geq 3$. This observation finally gives the expansion of f:
$$\langle (d\exp_p)_{tv} tw_1, (d\exp_p)_{tv} tw_2\rangle = \langle J_1(t), J_2(t)\rangle = f(t) = \frac{2\langle w_1, w_2\rangle_p}{2!}t^2 = \langle w_1, w_2\rangle_p t^2$$
which gives $\langle (d\exp_p)_v w_1, (d\exp_p)_v w_2\rangle_{\exp_p v} = \langle w_1, w_2\rangle_v$ and completes the proof.

Proof 2: Suppose $v \in B_\varepsilon(0) \subset T_pM$. We want to show, for $u_1, u_2 \in T_v(T_pM) = T_pM$, $\langle u_1, u_2 \rangle = \langle (d\exp_p)_v u_1, (d\exp_p)_v u_2 \rangle_{\exp_p v}$. By the Gauss lemma, it suffices to show this for $u_1, u_2 \perp v$. (We need this assumption to make sense of the "sectional curvature" of the plane spanned by $\{u_1, v\}$ or $\{u_2, v\}$.)

Let J be the Jacobi field along $\gamma(t) = \exp_p(tv)$ with $J(0) = 0$, $J'(0) = u_1$. Then $J'' = 0$ (see Example 2.3 on PP.112-113), so if U_1 is a parallel vector field along γ with $U_1(0) = u_1$, then $J(t) = tU_1(t)$. (To keep to the conventions used in Example 2.3, we may assume $|u_1| = |u_2| = 1$.) On the other hand, by Proposition 2.4 on PP.113 or Corollary 2.5 on PP.114 we know that $J(t) = (d\exp_p)_{tv}(tu_1)$. Thus $(d\exp_p)_v(u_1) = U_1(1)$. Similarly if U_2 is a parallel vector field along γ with $U_2(0) = u_2$, then $(d\exp_p)_v(u_2) = U_2(1)$. Thus

$$\langle (d\exp_p)_v(u_1), (d\exp_p)_v(u_2) \rangle_{\exp_p v} = \langle U_1(1), U_2(1) \rangle_{\exp_p v} = \langle u_1, u_2 \rangle_p$$

since parallel transport preserves inner product.

Remark: The only difference between Proof 1 and Proof 2 is that they utilize Example 2.3 to different levels. But the characterization of $R(\gamma', J)\gamma'$ for a Jacobi field J along a geodesic γ in a Riemannian manifold of constant sectional curvature is essential.

2. Let M be a Riemannian manifold, $\gamma : [0,1] \to M$ a geodesic, and J a Jacobi field along γ. Prove that there exists a parametrized surface $f(t,s)$, where $f(t,0) = \gamma(t)$ and the curves $t \to f(t,s)$ are geodesics, such that $J(t) = \frac{\partial f}{\partial s}(t,0)$.
Hint: Choose a curve $\lambda(s)$, $s \in (-\varepsilon, \varepsilon)$ in M such that $\lambda(0) = \gamma(0)$, $\lambda'(0) = J(0)$. Along λ choose a vector field $W(s)$ with $W(0) = \gamma'(0)$, $\frac{DW}{ds}(0) = \frac{DJ}{dt}(0)$. Define $f(s,t) = \exp_{\lambda(s)} tW(s)$ and verify that $\frac{\partial f}{\partial s}(0,0) = \frac{d\lambda}{ds}(0) = J(0)$ and

$$\frac{D}{dt}\frac{\partial f}{\partial s}(0,0) = \frac{D}{ds}\frac{\partial f}{\partial t}(0,0) = \frac{DW}{ds}(0) = \frac{DJ}{dt}(0).$$

- Proof: We follow the hint closely. The existence of $\lambda(s)$ is guaranteed by the existence (and uniqueness) of linear ODE systems. The existence of $W(s)$ follows from the same reason. Note that $f(t,0) = \exp_{\lambda(0)} tW(0) = \exp_{\gamma(0)} t\gamma'(0) = \gamma(t)$, and obviously that for each fixed $s \in (-\varepsilon,\varepsilon)$ the curve $t \mapsto f(t,s)$ is a geodesic. To see that $\frac{\partial f}{\partial s}(t,0)$ is indeed a geodesic, we compute directly

$$\frac{D}{dt}\frac{D}{dt}\frac{\partial f}{\partial s}(t,0) = \frac{D}{dt}\frac{D}{ds}\frac{\partial f}{\partial t}(t,0) = \frac{D}{ds}\left(\frac{D}{dt}\frac{\partial f}{\partial t}(t,0)\right) + R\left(\frac{\partial f}{\partial s}, \frac{\partial f}{\partial t}\right)\frac{\partial f}{\partial t}(t,0)$$

$$= 0 - R\left(\frac{\partial f}{\partial t}, \frac{\partial f}{\partial s}\right)\frac{\partial f}{\partial t}(t,0) = -R\left(\gamma'(t), \frac{\partial f}{\partial s}(t,0)\right)\gamma'(t).$$

By the uniqueness theorem of Jacobi fields along a given geodesic, it now suffices to show that $J(0) = \frac{\partial f}{\partial s}(0,0)$ and $\frac{DJ}{dt}(0) = \frac{D}{dt}\frac{\partial f}{\partial s}(0,0)$. In fact,

$$\frac{\partial f}{\partial s}(0,0) = \left.\frac{\partial f(0,s)}{\partial s}\right|_{s=0} = \left.\frac{\partial \lambda(s)}{\partial s}\right|_{s=0} = \lambda'(0) = J(0)$$

$$\frac{D}{dt}\frac{\partial f}{\partial s}(0,0) = \frac{D}{ds}\frac{\partial f}{\partial t}(0,0) = \left.\frac{D}{ds}\right|_{s=0}\left[\left(d\exp_{\lambda(s)}\right)_{tW(s)} W(s)\right]\Bigg|_{t=0} = \left.\frac{D}{ds}\right|_{s=0}\left(d\exp_{\lambda(s)}\right)_0 W(s)$$

$$= \frac{DW}{ds}(0) = \frac{DJ}{dt}(0)$$

which completes the proof.

3. Let M be a Riemannian manifold with non-positive sectional curvature. Prove that, for all p, the conjugate locus $C(p)$ is empty.

Hint: Assume the existence of a non-trivial Jacobi field along the geodesic $\gamma:[0,a]\longrightarrow M$, with $\gamma(0)=p$, $J(0)=J(a)=0$. Use the Jacobi equation to show that $\frac{d}{dt}\left\langle\frac{DJ}{dt},J\right\rangle\geq 0$. Conclude that $\left\langle\frac{DJ}{dt},J\right\rangle\equiv 0$. Since $\frac{d}{dt}\langle J,J\rangle=2\left\langle\frac{DJ}{dt},J\right\rangle\equiv 0$, we have $\|J\|^2=\text{const.}=0$, a contradiction.

- Proof: Assume the existence of a non-trivial Jacobi field along the geodesic $\gamma:[0,a]\to M$ with $\gamma(0)=p$, $J(0)=J(a)=0$. The Jacobi equation gives

$$\frac{d}{dt}\left\langle\frac{DJ}{dt},J\right\rangle=\left\langle\frac{D^2J}{dt^2},J\right\rangle+\left\langle\frac{DJ}{dt},\frac{DJ}{dt}\right\rangle=\left\langle -R(\gamma',J)\gamma',J\right\rangle+\left\langle\frac{DJ}{dt},\frac{DJ}{dt}\right\rangle\geq 0$$

since by hypothesis M has non-positive sectional curvature. But $J(0)=J(a)=0$ implies $\left\langle\frac{DJ}{dt},J\right\rangle(0)=\left\langle\frac{DJ}{dt},J\right\rangle(a)=0$, thus we must have $\frac{d}{dt}\left\langle\frac{DJ}{dt},J\right\rangle\equiv 0$.

Now we have $\frac{d}{dt}\langle J,J\rangle=2\left\langle\frac{DJ}{dt},J\right\rangle\equiv 0$ since $\left\langle\frac{DJ}{dt},J\right\rangle(0)=0$ and $\frac{d}{dt}\left\langle\frac{DJ}{dt},J\right\rangle\equiv 0$, from which we conclude that

$$\|J\|^2=\langle J,J\rangle=\text{const.}=\langle J,J\rangle(0)=0$$

contradicting our assumption that J is non-trivial.

4. Let $b<0$ and let M be a manifold with constant negative sectional curvature equal to b. Let $\gamma:[0,l]\longrightarrow M$ be a normalized geodesic, and let $v\in T_{\gamma(l)}M$ such that $\langle v,\gamma'(l)\rangle=0$ and $|v|=1$. Since M has negative curvature, $\gamma(l)$ is not conjugate to $\gamma(0)$ (see Exercise 3 above). Show that the Jacobi field J along γ determined by $J(0)=0$, $J(l)=v$ is given by

$$J(t)=\frac{\sinh(t\sqrt{-b})}{\sinh(l\sqrt{-b})}w(t),$$

where $w(t)$ is the parallel transport along γ of the vector

$$w(0)=\frac{u_0}{|u_0|},\quad u_0=(d\exp_p)^{-1}_{l\gamma'(0)}(v)$$

and where u_0 is considered as a vector in $T_{\gamma(0)}M$ by the identification $T_{\gamma(0)}M\approx T_{l\gamma'(0)}(T_{\gamma(0)}M)$.

Hint: From Example 2.3, the Jacobi field J_1 along γ satisfying $J_1(0)=0$, $J_1'(0)=\frac{u_0}{|u_0|}$, is given by $J_1(t) = \frac{\sinh t\sqrt{-b}}{\sqrt{-b}} \omega(t)$. In addition, from Corollary 2.5, $J_1(l) = (d\exp_p)_{l\gamma'(0)}(l\omega(0))$. It follows that

$$J(l) = v = (d\exp_p)_{l\gamma'(0)}(u_0) = J_1(l)\frac{|u_0|}{l}.$$

Therefore, $J(t) = J_1(t)\frac{|u_0|}{l} = \frac{\sinh t\sqrt{-b}}{\sqrt{-b}}\omega(t)\frac{|u_0|}{l}$.
In addition, since $1 = |v| = |J(l)| = \frac{\sinh l\sqrt{-b}}{\sqrt{-b}}\frac{|u_0|}{l}$, we have $\frac{|u_0|}{l} = \left(\frac{\sinh l\sqrt{-b}}{\sqrt{-b}}\right)^{-1}$, which implies what was asserted.

- Proof: We follow the hint. Assume $u_0 \in T_{l\gamma'(0)}(T_{\gamma(0)}M) \approx T_{\gamma(0)}M$, $\omega_0 = \frac{u_0}{|u_0|}$. By Example 2.3, the Jacobi field along geodesic γ satisfying $J_1(0) = 0$, $J_1'(0) = \omega_0$ in a manifold of constant negative curvature is given by $J_1(t) = \frac{\sinh(t\sqrt{-b})}{\sqrt{-b}}\omega(t)$ where $\omega(t)$ is the parallel transport of $\omega(0) = \omega_0$ along γ. On the other hand, by Corollary 2.5 we know $J(t) = (d\exp_p)_{t\gamma'(0)}(tJ'(0))$. Thus $v = J(l)$ implies
$$v = (d\exp_p)_{l\gamma'(0)}(lJ'(0)) = l \cdot (d\exp_p)_{l\gamma'(0)}(J'(0)) \Rightarrow J'(0) = \frac{1}{l}(d\exp_p)^{-1}_{l\gamma'(0)}(v) = \frac{u_0}{l} =$$
$$= \frac{u_0}{|u_0|}\cdot\frac{|u_0|}{l} = \frac{|u_0|}{l}\omega(0).$$

Note that $\frac{|u_0|}{l}J_1(t)$ is also a Jacobi field along geodesic $\gamma(t)$, satisfying $\frac{|u_0|}{l}J_1(0) = J(0) = 0$ and $\frac{D}{dt}\left(\frac{|u_0|}{l}J_1(t)\right)\Big|_{t=0} = \frac{|u_0|}{l}\frac{DJ_1}{dt}(0) = \frac{|u_0|}{l}\omega(0) = v = \frac{DJ}{dt}(0)$

Thus by the uniqueness we have $J(t) = \frac{|u_0|}{l}J_1(t) = \frac{|u_0|}{l}\frac{\sinh(t\sqrt{-b})}{\sqrt{-b}}\omega(t)$.
Letting $t = l$, we have $v = J(l) = \frac{|u_0|}{l}\frac{\sinh(l\sqrt{-b})}{\sqrt{-b}}\omega(l)$. Recall that
$1 = |v| = \frac{|u_0|}{l}\frac{\sinh(l\sqrt{-b})}{\sqrt{-b}} \Rightarrow \frac{|u_0|}{l} = \frac{\sqrt{-b}}{\sinh(l\sqrt{-b})}$. Hence we finally obtain
$$J(t) = \frac{\sqrt{-b}}{\sinh(l\sqrt{-b})}\frac{\sinh(t\sqrt{-b})}{\sqrt{-b}}\omega(t) = \frac{\sinh(t\sqrt{-b})}{\sinh(l\sqrt{-b})}\omega(t).$$

5. Jacobi fields and conjugate points on locally symmetric spaces (Cf. Exercise 6 of Chap 4). Let $\gamma:[0,\infty)\to M$ be a geodesic in a locally symmetric space M and let $v=\gamma'(0)$ be its velocity at $p=\gamma(0)$. Define a linear transformation $K_v: T_pM\to T_pM$ by

$$K_v(x)=R(v,x)v,\quad x\in T_pM.$$

a) Prove that K_v is self-adjoint.

b) Choose an orthonormal basis $\{e_1,\cdots,e_n\}$ of T_pM that diagonalizes K_v, that is, $K_v(e_i)=\lambda_i e_i$, $i=1,\cdots,n$. Extend the e_i to fields along γ by parallel transport. Show that, for all t, $K_{\gamma'(t)}(e_i(t))=\lambda_i e_i(t)$, where λ_i does not depend on t.

Hint: Use Exercise 6(a), of Chap. 4.

c) Let $J(t)=\sum_i x_i(t)e_i(t)$ be a Jacobi field along γ. Show that the Jacobi equation is equivalent to the system $\frac{d^2x_i}{dt^2}+\lambda_i x_i=0$, $i=1,\cdots,n$.

d) Show that the conjugate points of p along γ are given by $\gamma(\pi k/\sqrt{\lambda_i})$, where k is a positive integer and λ_i is a positive eigenvalue of K_v.

• $\boxed{a)}$ For any $y\in T_pM$, we have (for any $x\in T_pM$)

$$\langle x, K_v^*(y)\rangle_p=\langle K_v(x),y\rangle_p=\langle R(v,x)v,y\rangle_p=\langle R(v,y)v,x\rangle_p=\langle K_v(y),x\rangle_p=\langle x,K_v(y)\rangle_p$$

By the arbitrariness of $x\in T_pM$, this implies $K_v(y)=K_v^*(y)$ for all $y\in T_pM$, i.e. K_v is self-adjoint.

$\boxed{b)}$ Since $K_v: T_pM\to T_pM$ is self-adjoint, the Spectral Theorem gives the existence of an orthonormal basis $\{e_1,\cdots,e_n\}$ of T_pM that diagonalizes K_v, i.e. $K_v(e_i)=\lambda_i e_i$ for $i=1,\cdots,n$, where $\lambda_i\in\mathbb{R}$ for all $i\in\{1,\cdots,n\}$. Extend the e_i to fields along γ by parallel transport. Note that M is locally symmetric, by Exercise 6 a) in Chapter 4 (P.106) we know $K_{\gamma'(t)}(e_i(t))=R(\gamma'(t),e_i(t))\gamma'(t)$ is also a parallel vector field along γ. Since parallel transport preserves inner products, $\{e_i(t)\}$ forms a basis of $T_{\gamma(t)}M$ for each t. Thus we may write $K_{\gamma'(t)}(e_i(t))=\sum_{k=1}^n\lambda_k^i(t)e_k(t)$, where $\lambda_k^i(t)$ are smooth functions along γ. Since $\frac{D}{dt}K_{\gamma'(t)}(e_i(t))\equiv 0$, we have $\sum_{k=1}^n\frac{d\lambda_k^i}{dt}(t)e_k(t)=0$ for each t. By the linear independence of $\{e_k(t)\}$ at each t we get $\frac{d\lambda_k^i}{dt}\equiv 0$, i.e. $\lambda_k^i(t)=\text{const}$. Denote $\lambda_k^i(t)=\lambda_k^i$, then $K_{\gamma'(t)}(e_i(t))=\sum_{k=1}^n\lambda_k^i e_k(t)$.

At $t=0$, $\lambda_i e_i=K_v(e_i)=K_{\gamma'(0)}(e_i(0))=\sum_{k=1}^n\lambda_k^i e_k(0)=\sum_{k=0}^n\lambda_k^i e_k$. Again by the linear independence of $\{e_k\}$, we have $\lambda_k^i=\delta_k^i\lambda_k$, i.e. $K_{\gamma'(t)}(e_i(t))=\lambda_i e_i(t)$.

$\boxed{c)}$ $J(t)$ satisfies $J''+R(\gamma',J)\gamma'=0$, thus $\sum_{i=1}^n x_i''(t)e_i(t)+\sum_{i=1}^n R(\gamma'(t),x_i(t)e_i(t))\gamma'(t)=0$.

Note that for each $i \in \{1, \cdots, n\}$ we have $R(\gamma'(t), x_i(t)e_i(t))\gamma'(t) = x_i(t)R(\gamma'(t), e_i(t))\gamma'(t)$ $= x_i(t)K_{\gamma'(t)}(e_i(t)) = \lambda_i x_i(t) e_i(t)$. Thus $0 = \sum_{i=1}^{n}\left(x_i''(t)e_i(t) + R(\gamma'(t), x_i(t)e_i(t))\gamma'(t)\right) =$ $= \sum_{i=1}^{n}\left(x_i''(t) + \lambda_i x_i(t)\right)e_i(t)$. By the linear independence of $\{e_i(t)\}_{i=1}^{n}$, this implies

$$\frac{d^2 x_i}{dt^2} + \lambda_i x_i = 0 \quad \text{for all } i = 1, \cdots, n.$$

$\boxed{d)}$ Assume $q = \gamma(a)$ is a conjugate point of $p = \gamma(0)$ along γ. Then there exists a Jacobi field along γ such that $J(0) = J(a) = 0$ (obviously we require $a \neq 0$). This is further equivalent to the existence of a non-trivial solution to the following linear ODE system with boundary values:

$$\begin{cases} \dfrac{d^2 x_i}{dt^2} + \lambda_i x_i = 0, & i = 1, \cdots, n \\ x_i(0) = x_i(a) = 0, & i = 1, \cdots, n \end{cases}$$

For those λ_i which are nonpositive, the only solution to the boundary value problem $\begin{cases} \dfrac{d^2 x_i}{dt^2} + \lambda_i x_i = 0 \\ x_i(0) = x_i(a) = 0 \end{cases}$ is $x_i(t) \equiv 0$. For those λ_i which are positive, the problem has a non-trivial solution if and only if $\frac{k^2}{a^2}\pi^2 = \lambda_i \Leftrightarrow a = \pi k/\sqrt{\lambda_i}$, where $k \in \mathbb{Z}_{>0}$. (In this case, $x_i(t) = A\sin\left(\frac{k\pi}{a}t\right)$, where $A \in \mathbb{R}$ is some non-zero constant.)

Therefore, the conjugate points of p along γ are given by $\gamma(\pi k/\sqrt{\lambda_i})$, where $k \in \mathbb{Z}_{>0}$ and λ_i is a positive eigenvalue of K_v. (By the definition of conjugate points, we need to require $k \neq 0$. We need $k \geq 0$ because we parametrized $\gamma(t)$ in such a way that γ is defined for $t \geq 0$.)

6. Let M be a Riemannian manifold of dimension two (in this case we say that M is a surface). Let $B_\delta(p)$ be a normal ball around the point $p \in M$ and consider the parametrized surface $f(\rho,\theta) = \exp_p \rho v(\theta)$, $0<\rho<\delta$, $-\pi<\theta<\pi$, where $v(\theta)$ is a circle of radius 1 in T_pM parametrized by the central angle θ.

a) Show that (ρ,θ) are coordinates in an open set $\mathcal{U} \subset M$ formed by the open ball $B_\delta(p)$ minus the ray $\exp_p(-\rho v(0))$ $0\leq\rho<\delta$. Such coordinates are called <u>polar coordinates</u> at p.

b) Show that the coefficients g_{ij} of the Riemannian metric in these coordinates are: $g_{12}=0$, $g_{11} = \left|\frac{\partial f}{\partial \rho}\right|^2 = |v(\theta)|^2 = 1$, $g_{22} = \left|\frac{\partial f}{\partial\theta}\right|^2$.

c) Show that, along the geodesic $f(\rho,0)$, we have

$$\left(\sqrt{g_{22}}\right)_{\rho\rho} = -K(p)\rho + R(\rho), \text{ where } \lim_{\rho\to 0}\frac{R(\rho)}{\rho}=0 \text{ and } K(p) \text{ is the sectional curvature of } M \text{ at } p.$$

d) Prove that $\lim_{\rho\to 0}\frac{(\sqrt{g_{22}})_{\rho\rho}}{\sqrt{g_{22}}} = -K(p)$. This last expression is the value of the Gaussian curvature of M at p given in polar coordinates (Cf., for example, M. do Carmo [dC 2] p.288). This fact from the theory of surfaces and (d) shows that, in dimension two, the sectional curvature coincides with the Gaussian curvature. In the next chapter, we shall give a more direct proof of this fact.

$\boxed{a)}$ First note that $\exp_p : B_\delta(0) \longrightarrow B_\delta(p)$ is a diffeomorphism, thus $\exp_p \rho_1 v(\theta_1) = \exp_p \rho_2 v(\theta_2)$ if and only if $\rho_1 v(\theta_1) = \rho_2 v(\theta_2)$. Since $|v(\theta)| \equiv 1$, $\rho_1 v(\theta_1) = \rho_2 v(\theta_2)$ if and only if $\rho_1=\rho_2$, $v(\theta_1)=v(\theta_2)$. Note that θ is the central angle, thus if $-\pi<\theta_1,\theta_2<\pi$ then $v(\theta_1)=v(\theta_2)$ if and only if $\theta_1=\theta_2$. Hence $\rho v(\theta)$ gives an one-to-one correspondence between $(0,\delta)\times(-\pi,\pi)\subset\mathbb{R}^2$ and $V := B_\delta(0)\setminus\{\rho v(\pi) : 0\leq\rho<\delta\} \subset T_pM$. Further, note that $\exp_p|_V : V \longrightarrow \mathcal{U}$ is a diffeomorphism, thus $f(\rho,\theta) = \exp_p \rho v(\theta)$ gives a coordinate chart of the open set $\mathcal{U}$.

$\boxed{b)}$ A basis of the tangent space on M at $f(\rho_0,\theta_0)$ can be chosen as $\left\{\frac{\partial}{\partial\rho}\Big|_{(\rho_0,\theta_0)}, \frac{\partial}{\partial\theta}\Big|_{(\rho_0,\theta_0)}\right\}$. By definition, they are derivatives of certain curves on M. Specifically, let γ_1, γ_2 be curves lying in a neighborhood of $f(\rho_0,\theta_0)$ such that $\gamma_1(t) = \exp_p[(\rho_0+t)v(\theta_0)]$, $\gamma_2(t) = \exp_p[\rho_0 v(\theta_0+t)]$. Then $\frac{\partial}{\partial\rho}\Big|_{(\rho_0,\theta_0)} = \gamma_1'(0) = (d\exp_p)_{\rho_0 v(\theta_0)}(v(\theta_0))$, $\frac{\partial}{\partial\theta}\Big|_{(\rho_0,\theta_0)} = \gamma_2'(0) = (d\exp_p)_{\rho_0 v(\theta_0)}(\rho_0 v'(\theta_0))$.
Since $|v(\theta)|\equiv 1$, we have $\langle v(\theta), v'(\theta)\rangle \equiv 0$ for all $\theta\in(-\pi,\pi)$. Now by the Gauss Lemma

$$g_{12} = \left\langle \frac{\partial}{\partial \rho}\Big|_{(\rho_0,\theta_0)}, \frac{\partial}{\partial \theta}\Big|_{(\rho_0,\theta_0)} \right\rangle = \left\langle (d\exp_p)_{\rho_0 v(\theta_0)}(v(\theta_0)), (d\exp_p)_{\rho_0 v(\theta_0)}(\rho_0 v'(\theta_0)) \right\rangle$$

$$= \frac{1}{\rho_0}\left\langle (d\exp_p)_{\rho_0 v(\theta_0)}(\rho_0 v(\theta_0)), (d\exp_p)_{\rho_0 v(\theta_0)}(\rho_0 v'(\theta_0)) \right\rangle = \frac{1}{\rho_0}\langle \rho_0 v(\theta_0), \rho_0 v'(\theta_0)\rangle$$

$$= \rho_0 \langle v(\theta_0), v'(\theta_0)\rangle = 0$$

$$g_{11} = \left\langle \frac{\partial}{\partial \rho}\Big|_{(\rho_0,\theta_0)}, \frac{\partial}{\partial \rho}\Big|_{(\rho_0,\theta_0)} \right\rangle = \left|\frac{\partial f}{\partial \rho}\right|^2\Bigg|_{(\rho_0,\theta_0)} = \left\langle (d\exp_p)_{\rho_0 v(\theta_0)}(v(\theta_0)), (d\exp_p)_{\rho_0 v(\theta_0)}(v(\theta_0)) \right\rangle$$

$$= \frac{1}{\rho_0^2}\left\langle (d\exp_p)_{\rho_0 v(\theta_0)}(\rho_0 v(\theta_0)), (d\exp_p)_{\rho_0 v(\theta_0)}(\rho_0 v(\theta_0)) \right\rangle = \frac{1}{\rho_0^2}\langle \rho_0 v(\theta_0), \rho_0 v(\theta_0)\rangle$$

$$= \langle v(\theta_0), v(\theta_0)\rangle = 1$$

$$g_{22} = \left\langle \frac{\partial}{\partial \theta}\Big|_{(\rho_0,\theta_0)}, \frac{\partial}{\partial \theta}\Big|_{(\rho_0,\theta_0)} \right\rangle = \left|\frac{\partial f}{\partial \theta}\right|^2\Bigg|_{(\rho_0,\theta_0)}.$$

c) Let $J(\rho) = \frac{\partial f}{\partial \theta}(\rho,0) = (d\exp_p)_{\rho v(0)}(\rho v(0))$. Then it is easy to see that $J(\rho)$ is a Jacobi field along the geodesic $\rho \longmapsto \exp_p(\rho v(0))$ with $J(0)=0$, $\left|\frac{DJ}{d\rho}(0)\right| = 1$. By Corollary 2.10, we know that $\sqrt{g_{22}} = \left|\frac{\partial f}{\partial \theta}(\rho,0)\right| = |J(\rho)| = \rho - \frac{1}{6}K(p)\rho^3 + o(\rho^3)$ where we used the fact $\langle v(0), v'(0)\rangle = 0$ to ensure that the sectional curvature at p of the plane spanned by $\{v(0), v'(0)\}$ makes sense. It follows immediately that $\left(\sqrt{g_{22}}\right)_{\rho\rho} = -K(p)\rho + o(\rho)$. Since M is a surface, the notation $K(p)$ makes sense.

d) $$\lim_{\rho\to 0} \frac{(\sqrt{g_{22}})_{\rho\rho}}{\sqrt{g_{22}}} = \lim_{\rho\to 0}\frac{-K(p)\rho + o(\rho)}{\rho - \frac{1}{6}K(p)\rho^3 + o(\rho^3)} = \lim_{\rho\to 0}\frac{-K(p) + o(1)}{1 - \frac{1}{6}K(p)\rho^2 + o(\rho^2)} = -K(p).$$

Remark. In this problem we take for granted that the Riemannian metric of a surface is induced from the ambient Euclidean space with the canonical Euclidean metric.

7. Let M be a Riemannian manifold of dimension two. Let $p\in M$ and let $V\subset T_pM$ be a neighborhood of the origin where $\exp_p$ is a diffeomorphism. Let $S_r(0)\subset V$ be a circle of radius r centered at the origin, and let L_r be the length of the curve $\exp_p(S_r)$ in M. Prove that the sectional curvature at $p\in M$ is given by $K(p)=\lim_{r\to 0}\frac{3}{\pi}\frac{2\pi r-L_r}{r^3}$.

Hint: Use Exercise 6.

- Proof: Following the notations used in Exercise 6, we have $\exp_p(S_r)=\exp_p(rv(\theta))$, $-\pi<\theta<\pi$.
Thus $L_r=\int_{-\pi}^{\pi}\left|\frac{\partial f}{\partial\theta}(r,\theta)\right|d\theta=\int_{-\pi}^{\pi}\sqrt{g_{22}}\,d\theta$ where $f(r,\theta)=\exp_p rv(\theta)$, $0<r<\delta$, $-\pi<\theta<\pi$.
As we have seen in Solution to Exercise 6 c), there holds for each fixed $\theta\in(-\pi,\pi)$

$$\sqrt{g_{22}}=r-\frac{1}{6}K(p)r^3+R(r,\theta)\quad\text{where}\quad\lim_{r\to 0}\frac{R(r,\theta)}{r^3}=0.$$

Hence $L_r=\int_{-\pi}^{\pi}\sqrt{g_{22}}\,d\theta=\int_{-\pi}^{\pi}\left(r-\frac{1}{6}K(p)r^3+R(r,\theta)\right)d\theta=2\pi r-\frac{1}{3}K(p)\cdot\pi r^3+\int_{-\pi}^{\pi}R(r,\theta)d\theta$, which gives

$$K(p)=\frac{3}{\pi r^3}\left(2\pi r-L_r+\int_{-\pi}^{\pi}R(r,\theta)\,d\theta\right).$$

Since the left hand side is independent of r, we can take limit as $r\to 0$ on the right hand side. Note that $\lim_{r\to 0}\frac{R(r,\theta)}{r^3}=0$ uniformly in θ, we can commute the limit process with integration to conclude that

$$K(p)=\lim_{r\to 0}\frac{3}{\pi r^3}\left(2\pi r-L_r+\int_{-\pi}^{\pi}R(r,\theta)\,d\theta\right)=\lim_{r\to 0}\frac{3}{\pi}\frac{2\pi r-L_r}{r^3}+\lim_{r\to 0}\frac{3}{\pi}\int_{-\pi}^{\pi}\frac{R(r,\theta)}{r^3}d\theta$$

$$=\lim_{r\to 0}\frac{3}{\pi}\frac{2\pi r-L_r}{r^3}+\frac{3}{\pi}\int_{-\pi}^{\pi}\left(\lim_{r\to 0}\frac{R(r,\theta)}{r^3}\right)d\theta=\lim_{r\to 0}\frac{3}{\pi}\frac{2\pi r-L_r}{r^3}+0=\lim_{r\to 0}\frac{3}{\pi}\frac{2\pi r-L_r}{r^3}.$$

8. Let $\gamma:[0,a]\to M$ be a geodesic and let X be a Killing field on M.
a) Show that the restriction $X(\gamma(s))$ of X to $\gamma(s)$ is a Jacobi field along γ.
b) Use item (a) to show that (cf. Exercise 6 of Chap. 3) if M is connected and there exists $p\in M$ with $X(p)=0$ and $\nabla_Y X(p)=0$, for all $Y(p)\in T_pM$, then $X=0$ on M.

- $\boxed{a)}$ Consider the flow $\varphi(t,q)$ of X defined by $\begin{cases}\frac{\partial}{\partial t}\varphi(t,q)=X(\varphi(t,q)) & t\in(-\varepsilon,\varepsilon)\\ \varphi(0,q)=q\end{cases}$
Since X is a Killing field, $q\longmapsto\varphi(t,q)$ is an isometry for each $t\in(-\varepsilon,\varepsilon)$. Hence $s\longmapsto\varphi(t,\gamma(s))$ is a geodesic for each $t\in(-\varepsilon,\varepsilon)$.
Denote $f(t,s):=\varphi(t,\gamma(s))$, then $f(0,s)=\varphi(0,\gamma(s))=\gamma(s)$, $\frac{\partial f}{\partial t}(0,s)=\frac{\partial}{\partial t}\Big|_{t=0}\varphi(t,\gamma(s))$
$=X(\varphi(0,\gamma(s)))=X(\gamma(s))$.

Note that $\frac{D}{ds}\frac{\partial f}{\partial s}(t,s) \equiv 0$, thus $\frac{D}{dt}\frac{D}{ds}\frac{\partial f}{\partial s}(t,s) \equiv 0$, and we have

$$0 = \frac{D}{dt}\frac{D}{ds}\frac{\partial f}{\partial s}(0,s) = \frac{D}{ds}\frac{D}{dt}\frac{\partial f}{\partial s}(0,s) + \left(R\left(\frac{\partial f}{\partial s}, \frac{\partial f}{\partial t}\right)\frac{\partial f}{\partial s}\right)(0,s)$$

$$= \frac{D}{ds}\frac{D}{ds}\frac{\partial f}{\partial t}(0,s) + R\left(\frac{\partial f}{\partial s}(0,s), \frac{\partial f}{\partial t}(0,s)\right)\frac{\partial f}{\partial s}(0,s)$$

$$= \frac{D^2}{ds^2}X(\gamma(s)) + R(\gamma'(s), X(\gamma(s)))\gamma'(s)$$

where we used $\frac{\partial f}{\partial s}(0,s) = \frac{\partial (f(0,s))}{\partial s} = \frac{\partial(\varphi(0,\gamma(s)))}{\partial s} = \frac{\partial \gamma(s)}{\partial s} = \gamma'(s)$.

Thus $X(\gamma(s))$ satisfies the Jacobi equation along geodesic $\gamma(s)$; in other words, $X(\gamma(s))$ is a Jacobi field along γ.

$\boxed{b)}$ Let $B_\varepsilon(p)$ be a geodesic normal ball centered at $p \in M$. Let $q \in B_\varepsilon(p)$, $q \neq p$, and let $\gamma : [0,a] \longrightarrow M$ be a geodesic connecting p to q. From a) we know the restriction of X to γ is a Jacobi field along γ. By our assumption, $X(\gamma(0)) = X(p) = 0$ and $\frac{D(X\circ\gamma)}{ds}(0) = (\nabla_{\gamma'}X)(\gamma(0)) = (\nabla_{\gamma'}X)(p) = 0$. Thus $X(\gamma(s)) \equiv 0$ by the uniqueness of Jacobi fields with prescribed initial conditions. In particular, $X(q) = X(\gamma(a)) = 0$. By the arbitrariness of $q \in B_\varepsilon(p)\setminus\{p\}$, we know $X \equiv 0$ in $B_\varepsilon(p)$. Similarly, we can show that the set $\mathcal{U} := \{q \in M : X(q) = 0\}$ is open in M. On the other hand, obviously $\mathcal{U}$ is closed since X is smooth. Thus it follows from the connectedness of M that $\mathcal{U} = M$, or equivalently $X = 0$ on M.

No.

Date

1. Let M_1 and M_2 be Riemannian manifolds, and consider the product $M_1 \times M_2$, with the product measure. Let ∇^1 be the Riemannian connection of M_1 and let ∇^2 be the Riemannian connection of M_2.

a) Show that the Riemannian connection ∇ of $M_1 \times M_2$ is given by

$$\nabla_{Y_1+Y_2}(X_1+X_2) = \nabla^1_{Y_1}X_1 + \nabla^2_{Y_2}X_2, \quad X_1, Y_1 \in \mathfrak{X}(M_1),\ X_2, Y_2 \in \mathfrak{X}(M_2).$$

b) For every $p \in M_1$, the set $(M_2)_p = \{(p,q) \in M_1 \times M_2 ;\ q \in M_2\}$ is a submanifold of $M_1 \times M_2$, naturally diffeomorphic to M_2. Prove that $(M_2)_p$ is a totally geodesic submanifold of $M_1 \times M_2$.

c) Let $\sigma(x,y) \subset T_{(p,q)}(M_1 \times M_2)$ be a plane such that $x \in T_pM_1$ and $y \in T_qM_2$. Show that $K(\sigma) = 0$.

- Proof: a) It is direct to verify by definition that $\nabla : \mathfrak{X}(M_1 \times M_2) \times \mathfrak{X}(M_1 \times M_2) \longrightarrow \mathfrak{X}(M_1 \times M_2)$ is a connection on $M_1 \times M_2$. By the uniqueness of a Levi-Civita connection on a Riemannian manifold, it suffices to show that ∇ is torsion free and compatible with the product measure.

For all $X_1, Y_1 \in \mathfrak{X}(M_1)$, $X_2, Y_2 \in \mathfrak{X}(M_2)$, one has $[X_1, Y_2] = 0 = [X_2, Y_1]$ by definition.

Thus $\nabla_{Y_1+Y_2}(X_1+X_2) - \nabla_{X_1+X_2}(Y_1+Y_2) = \nabla^1_{Y_1}X_1 + \nabla^2_{Y_2}X_2 - \nabla^1_{X_1}Y_1 - \nabla^2_{X_2}Y_2 = [Y_1, X_1] + [Y_2, X_2] =$

$= [Y_1, X_1] + [Y_2, X_1] + [Y_1, X_2] + [Y_2, X_2] = [Y_1+Y_2, X_1+X_2]$. cf. Exercise 14, Chapter 3 Solution Part 4°.

For all $X_1, Y_1, Z_1 \in \mathfrak{X}(M_1)$, $X_2, Y_2, Z_2 \in \mathfrak{X}(M_2)$, we have $\nabla_{Z_i}X_j = 0 = \nabla_{Z_i}Y_j$ whenever $i \neq j$.

$(Z_1+Z_2)g(X_1+X_2, Y_1+Y_2) = (Z_1+Z_2)(g_1(X_1,Y_1) + g_2(X_2,Y_2)) = Z_1 g_1(X_1,Y_1) + Z_2 g_2(X_2,Y_2) =$

$= g_1(\nabla^1_{Z_1}X_1, Y_1) + g_1(X_1, \nabla^1_{Z_1}Y_1) + g_2(\nabla^2_{Z_2}X_2, Y_2) + g_2(X_2, \nabla^2_{Z_2}Y_2)$

$= g(\nabla^1_{Z_1}X_1 + \nabla^2_{Z_2}X_2, Y_1+Y_2) + g(X_1+X_2, \nabla^1_{Z_1}Y_1 + \nabla^2_{Z_2}Y_2)$

$= g(\nabla_{Z_1+Z_2}(X_1+X_2), Y_1+Y_2) + g(X_1+X_2, \nabla_{Z_1+Z_2}(Y_1+Y_2))$

Hence ∇ as defined in the problem is the Riemannian connection of $M_1 \times M_2$.

b) For any $X, Y \in T_{(p,q)}(M_2)_p$, we have

$B(X,Y) = \nabla_X Y - \nabla^2_X Y = \nabla^1_0 0 + \nabla^2_X Y - \nabla^2_X Y = 0$ (where $q \in M_2$ is arbitrary)

Thus $(M_2)_p$ is a totally geodesic submanifold of $M_1 \times M_2$.

c) Let $X \in \mathfrak{X}(M_1)$ be a local extension of $x \in T_pM_1$, and $Y \in \mathfrak{X}(M_2)$ a local extension of $y \in T_qM_2$.

Then $K(\sigma) = \langle R(X,Y)X, Y\rangle(p,q) = \langle \nabla_Y\nabla_X X - \nabla_X \cancel{\nabla_Y X}^{0} + \nabla_{\cancel{[X,Y]}^{0}} X, Y\rangle(p,q) = \langle \nabla^2_Y(\nabla^1_X X), Y\rangle(p,q) = 0.$

2. Show that $\vec{x}:\mathbb{R}^2 \longrightarrow \mathbb{R}^4$ given by

$$\vec{x}(\theta,\varphi)=\frac{1}{\sqrt{2}}(\cos\theta,\sin\theta,\cos\varphi,\sin\varphi),\quad (\theta,\varphi)\in\mathbb{R}^2$$

is an immersion of $\mathbb{R}^2$ into the unit sphere $S^3(1)\subset\mathbb{R}^4$, whose image $\vec{x}(\mathbb{R}^2)$ is a torus T^2 with sectional curvature zero in the induced metric.

- Proof: It is obvious to see that $\vec{x}(\mathbb{R}^2)\subset S^3(1)$. To see $\vec{x}:\mathbb{R}^2\longrightarrow\mathbb{R}^4$ is an immersion, we compute the Jacobi of $\vec{x}$:

$$J_{(\theta,\varphi)}\vec{x}=\frac{1}{\sqrt{2}}\begin{bmatrix}-\sin\theta & \cos\theta & 0 & 0\\ 0 & 0 & -\sin\varphi & \cos\varphi\end{bmatrix}$$

from which we deduce $\ker J_{(\theta,\varphi)}\vec{x}=\{0\}$ and $\operatorname{rank} J_{(\theta,\varphi)}\vec{x}=2$. Since the metric of $S^3(1)$ is induced from the Euclidean metric on $\mathbb{R}^4$, we know from the injectivity of $i_1: T_{\vec{x}(\theta,\varphi)}S^3(1)\hookrightarrow T_{\vec{x}(\theta,\varphi)}\mathbb{R}^4$ and $i_2=i_1\circ\vec{x}: T_{(\theta,\varphi)}\mathbb{R}^2\hookrightarrow T_{\vec{x}(\theta,\varphi)}\mathbb{R}^4$ that $\vec{x}: T_{(\theta,\varphi)}\mathbb{R}^2\longrightarrow T_{\vec{x}(\theta,\varphi)}S^3(1)$ is injective at each $(\theta,\varphi)\in\mathbb{R}^2$, i.e. $\vec{x}:\mathbb{R}^2\longrightarrow S^3(1)$ is an immersion.

$$\begin{tikzcd} T_{(\theta,\varphi)}\mathbb{R}^2 \arrow[r, "\vec{x}"] \arrow[rr, bend right, "i_2"'] & T_{\vec{x}(\theta,\varphi)}S^3(1) \arrow[r, hook, "i_1"] & \mathbb{R}^4 \end{tikzcd}$$

That $\vec{x}(\mathbb{R}^2)$ is a torus T^2 can be shown in a way similar to Exercise 1 b) of Chapter 1, for which we first check $\vec{x}(\mathbb{R}^2)=S^1\times S^1$ easily by definition. (It is also reasonable to take for granted that $T^2=S^1\times S^1$ by definition, as do Carmo did on P.42 Example 2.7.) Note that the metric on $S^3(1)$ we were using is induced from the canonical Euclidean metric on $\mathbb{R}^4$, we can think of the metric on $\vec{x}(\mathbb{R}^2)=S^1\times S^1$ also as the one induced from $\mathbb{R}^4$, i.e. the product metric on $S^1\times S^1$. Since the only possible non-zero sectional curvature on $S^1\times S^1$ is $K(\sigma)$, where $\sigma=\operatorname{span}\{x,y\}$, $x\in T_{\pi_1(p,q)}(\pi_1(S^1))$, $y\in T_{\pi_2(p,q)}(\pi_2(S^1))$, $(p,q)\in S^1\times S^1$, we obtain from Exercise 1 c) of Chapter 6 that $\vec{x}(\mathbb{R}^2)=S^1\times S^1$ has sectional curvature zero in the induced metric, as the $K(\sigma)$ mentioned above is identically zero by that Exercise (which we did on the previous page). Here $\pi_1: S^1\times S^1\longrightarrow S^1$ and $\pi_2: S^1\times S^1\longrightarrow S^1$ are the natural projections onto the first and second component respectively, as described in Example 2.7 on P.42.

3. Let M be a Riemannian manifold and let $N \subset K \subset M$ be submanifolds of M. Suppose that N is totally geodesic in K and that K is totally geodesic in M. Prove that N is totally geodesic in M.

Proof: If X and Y are local vector fields on N, we let X^K, Y^K be their arbitrary extensions to local vector fields on K, and X^M, Y^M be arbitrary extensions of X^K, Y^K to local vector fields on M. By definition, X^M, Y^M are also extensions of X, Y to local vector fields on M respectively. According to our hypotheses, $\nabla^M_{X^M} Y^M - \nabla^N_X Y = (\nabla^M_{X^M} Y^M - \nabla^K_{X^K} Y^K) + (\nabla^K_{X^K} Y^K - \nabla^N_X Y) = 0 + 0 = 0$. Thus N is totally geodesic in M.

4. Let $N_1 \subset M_1$, $N_2 \subset M_2$ be totally geodesic submanifolds of the Riemannian manifolds M_1 and M_2, respectively. Prove that $N_1 \times N_2$ is a totally geodesic submanifold of the product $M_1 \times M_2$ with the product metric.

Proof: Under the most naturally chosen notations, for any $X_1, Y_1 \in \mathfrak{X}(N_1)$, $X_2, Y_2 \in \mathfrak{X}(N_2)$ one has $B(X_1+X_2, Y_1+Y_2) = \overline{\nabla}_{\bar{X}_1+\bar{X}_2}(\bar{Y}_1+\bar{Y}_2) - \nabla_{X_1+X_2}(Y_1+Y_2) = \overline{\nabla}^1_{\bar{X}_1}\bar{Y}_1 + \overline{\nabla}^2_{\bar{X}_2}\bar{Y}_2 - \nabla^1_{X_1}Y_1 - \nabla^2_{X_2}Y_2 =$ $= (\overline{\nabla}^1_{\bar{X}_1}\bar{Y}_1 - \nabla^1_{X_1}Y_1) + (\overline{\nabla}^2_{\bar{X}_2}\bar{Y}_2 - \nabla^2_{X_2}Y_2) = B^1(X_1, Y_1) + B^2(X_2, Y_2) = 0 + 0 = 0$ by our hypotheses (cf. Solution to Exercise 1 a) of Chapter 6 for some of the notations used here.)

5. Prove that the sectional curvature of the Riemannian manifold $S^2 \times S^2$ with the product metric, where S^2 is the unit sphere in $\mathbb{R}^3$, is non-negative. Find a totally geodesic, flat torus, T^2, embedded in $S^2 \times S^2$.

Proof: Let x, y be linearly independent tangent vectors in $T_{(p,q)}(S^2 \times S^2)$. If x, y belong to tangent spaces of $\pi_1(S^1 \times S^1)$ and $\pi_2(S^1 \times S^1)$ respectively, then $R(x,y,x,y) = 0$ by Exercise 1 c) of Chapter 6 (the current chapter); if x, y both belong to the tangent space of $\pi_1(S^1 \times S^1)$ or that of $\pi_2(S^1 \times S^1)$, then $R(x,y,x,y) = 1$ since S^1 with the round metric has constant sectional curvature 1. Hence $S^2 \times S^2$ with the product metric has non-negative sectional curvature.

Note that a great circle S^1 in S^2 is totally geodesic by Proposition 2.9 on P.132, thus $T^2 = S^1 \times S^1$ is a totally geodesic embedded submanifold in $S^2 \times S^2$ by Exercise 4 of Chapter 6 (the previous exercise). That this torus is the flat torus follows from the definition of a flat torus on P.42 in Example 2.7, and the observation that the metric on $S^1 \subset S^2$ induced from the round metric on S^2 is indeed the round metric on S^1 (which is induced from the canonical Euclidean metric on $\mathbb{R}^2$).

No.

Date

6. Let G be a Lie group with a bi-invariant metric. Let H be a Lie group and let $h: H \longrightarrow G$ be an immersion that is also a homomorphism of groups (that is, H is a Lie subgroup of G). Show that h is a totally geodesic immersion.

- Proof: Denote $g(\,,\,)$ for the bi-invariant metric on G. The pull-back metric h^*g is (also) a bi-invariant metric on H, since for any $x, y \in H$ and $u, v \in T_xH$

$$(h^*g)((dL_x)_y u, (dL_x)_y v)(L_x(y)) = g(dh_{L_x(y)}((dL_x)_y u),\ dh_{L_x(y)}((dL_x)_y v))(h(L_x(y))) =$$

$$= g\Big(d(h \circ L_x)_y u,\ d(h \circ L_x)_y v\Big)(h(L_x(y))) = g\Big(d(L_{h(x)} \circ h)_y u,\ d(L_{h(x)} \circ h)_y v\Big)(L_{h(x)}(h(y)))$$

$$= g\Big(d(L_{h(x)})_{h(y)}(dh_y(u)),\ d(L_{h(x)})_{h(y)}(dh_y(v))\Big)(L_{h(x)}(h(y)))$$

$$= g\Big(dh_y(u),\ dh_y(v)\Big)(h(y)) = (h^*g)(u,v)(y)$$

where we used $(h \circ L_x)(y) = h(xy) = h(x)h(y) = (L_{h(x)} \circ h)(y)$. This proves that h^*g is left-invariant. Similarly, noting that $(h \circ R_x)(y) = h(yx) = h(y)h(x) = (R_{h(x)} \circ h)(y)$, we can show that h^*g is right-invariant in a completely parallel manner.

Now we have a bi-invariant metric g on G and a bi-invariant metric h^*g on H. Recall from Exercise 3 b) of Chapter 3 that if a Lie group $\tilde{G}$ has a bi-invariant metric then the geodesics of $\tilde{G}$ that start from $e \in \tilde{G}$ are 1-parameter subgroups of $\tilde{G}$. This immediately tells us that any geodesic γ in H starting from $e_H \in H$ is mapped to a geodesic $h \circ \gamma$ in G starting from $e_G \in G$. Indeed, since H, G are both equipped with bi-invariant metrics, γ is a 1-parameter subgroup of H, and then $h \circ \gamma$ is a 1-parameter subgroup of G since h is a Lie group homomorphism, which implies $h \circ \gamma$ is a geodesic in G starting from $e_G = h(e_H)$. Now, for any geodesic γ in H starting at $x \in H$, we claim that $L_{x^{-1}} \circ \gamma$ is a geodesic in H starting at e_H, and similarly $h \circ \gamma = (h \circ L_x)(L_{x^{-1}} \circ \gamma) = (L_{h(x)} \circ h)(L_{x^{-1}} \circ \gamma) = L_{h(x)}((h \circ L_{x^{-1}})(\gamma))$ is a geodesic starting at $h(x) \in G$. Once this is done, by Proposition 2.9 on P132 we can conclude that $h: H \longrightarrow G$ is totally geodesic since we showed that every geodesic γ of H starting from $x \in H$ is mapped to a geodesic of G at $h(x) \in G$. Finally, to verify the claim, note that the left-invariance of the metrics g; h^*g implies that $L_{x^{-1}}$ is a local isometry on H and $L_{h(x)}$ is a local isometry on G, and recall that local isometries locally preserves geodesics. This completes the whole proof.

7. Show that if M is a totally geodesic submanifold of $\overline{M}$, then, for any tangent fields to M, ∇ and $\overline{\nabla}$ coincide.

- Proof: M is totally geodesic in $\overline{M} \Leftrightarrow$ at all $p \in M$, for all $\eta \in (T_pM)^\perp$ the second fundamental form H_η is identically zero at p, i.e. $H_\eta(x,x) = 0$ for all $x \in T_pM$. We observe that this implies $H_\eta(x,y) = 0$ for all $x, y \in T_pM$ and all $\eta \in (T_pM)^\perp$:
$0 = H_\eta(x+y, x+y) = H_\eta(x,x) + H_\eta(x,y) + H_\eta(y,x) + H_\eta(y,y) = 2H_\eta(x,y) \Rightarrow H_\eta(x,y) = 0$.
By definition of H_η, this gives $\langle B(x,y), \eta \rangle = 0$ for all $x, y \in T_pM$, all $\eta \in (T_pM)^\perp$.
Thus $0 \equiv B(x,y) = \overline{\nabla}_{\bar{x}} \bar{y} - \nabla_x y$ for all $x, y \in T_pM$, i.e. ∇ and $\overline{\nabla}$ coincide for all tangent fields to M.

8. (The Clifford Torus). Consider the immersion $\vec{x}: \mathbb{R}^2 \to \mathbb{R}^4$ given in Exercise 2.
a) Show that the vectors $e_1 = (-\sin\theta, \cos\theta, 0, 0)$, $e_2 = (0, 0, -\sin\varphi, \cos\varphi)$ form an orthonormal basis of the tangent space, and that the vectors $n_1 = \frac{1}{\sqrt{2}}(\cos\theta, \sin\theta, \cos\varphi, \sin\varphi)$, $n_2 = \frac{1}{\sqrt{2}}(-\cos\theta, -\sin\theta, \cos\varphi, \sin\varphi)$ form an orthonormal basis of the normal space.
b) Use the fact that $\langle S_{n_k}(e_i), e_j \rangle = -\langle \overline{\nabla}_{e_i} n_k, e_j \rangle = \langle \overline{\nabla}_{e_i} e_j, n_k \rangle$, where $\overline{\nabla}$ is the covariant derivative (that is, the usual derivative) of $\mathbb{R}^4$, and $i, j, k = 1, 2$, to establish that the matrices of S_{n_1} and S_{n_2} with respect to the basis $\{e_1, e_2\}$ are $S_{n_1} = \begin{pmatrix} -1 & 0 \\ 0 & -1 \end{pmatrix}$, $S_{n_2} = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$

c) From Exercise 2, $\vec{x}$ is an immersion of the torus T^2 into $S^3(1)$ (the Clifford torus). Show that $\vec{x}$ is a minimal immersion.

- Proof: a) Direct computation gives
$$\frac{\partial \vec{x}}{\partial \theta} = \frac{1}{\sqrt{2}}(-\sin\theta, \cos\theta, 0, 0), \quad \frac{\partial \vec{x}}{\partial \varphi} = \frac{1}{\sqrt{2}}(0, 0, -\sin\varphi, \cos\varphi)$$
Thus an orthonormal basis of the tangent space is given by
$$e_1 = \frac{\partial \vec{x}}{\partial \theta} \Big/ \left\|\frac{\partial \vec{x}}{\partial \theta}\right\| = (-\sin\theta, \cos\theta, 0, 0), \quad e_2 = \frac{\partial \vec{x}}{\partial \varphi} \Big/ \left\|\frac{\partial \vec{x}}{\partial \varphi}\right\| = (0, 0, -\sin\varphi, \cos\varphi).$$
Further computation gives $n_1, n_2 \in \mathrm{span}_{\mathbb{R}}\{e_1, e_2\}^\perp$, $\langle n_1, n_2 \rangle = 0$, $\langle n_1, n_1 \rangle = 1$, $\langle n_2, n_2 \rangle = 1$. Thus n_1, n_2 form an orthonormal basis of the normal space.
b) Since $\mathbb{R}^4$ is of constant sectional curvature zero, all Christoffel symbols vanish. In order to compute $\overline{\nabla}_{e_1} e_1$ at $\vec{x}(\theta, \varphi) = \frac{1}{\sqrt{2}}(\cos\theta, \sin\theta, \cos\varphi, \sin\varphi) \in \mathbb{R}^4$, first note that the tangent vector field along $\vec{x}$ given by $e_1: (\theta, \varphi) \mapsto (-\sin\theta, \cos\theta, 0, 0) \in T_{\vec{x}(\theta,\varphi)}\mathbb{R}^4$ can

No.

Date

be smoothly extended to a vector field on $\mathbb{R}^4$ given by

$$\vec{e}_1 : \mathbb{R}^4 \ni (x_1, x_2, x_3, x_4) \longmapsto \sqrt{2}\,(-x_2, x_1, 0, 0) = \sqrt{2}\left(-x_2\frac{\partial}{\partial x_1} + x_1\frac{\partial}{\partial x_2}\right)$$

thus $\bar{\nabla}_{e_1} e_1 = \bar{\nabla}_{\vec{e}_1} \vec{e}_1 = \sum_{k=1}^{4} \vec{e}_1(E_k)\frac{\partial}{\partial x_k} = -(\sqrt{2})^2 x_1 \frac{\partial x_2}{\partial x_2}\frac{\partial}{\partial x_1} - (\sqrt{2})^2 x_2 \frac{\partial x_1}{\partial x_1}\frac{\partial}{\partial x_2} = -2\left(x_1\frac{\partial}{\partial x_1} + x_2\frac{\partial}{\partial x_2}\right)$

Similarly, $e_2 : (\theta, \varphi) \longmapsto (0, 0, -\sin\varphi, \cos\varphi) \in T_{\chi(\theta,\varphi)}\mathbb{R}^4$ can be smoothly extended to a vector field on $\mathbb{R}^4$ given by

$$\vec{e}_2 : \mathbb{R}^4 \ni (x_1, x_2, x_3, x_4) \longmapsto \sqrt{2}\,(0, 0, -x_4, x_3) = \sqrt{2}\left(-x_4\frac{\partial}{\partial x_3} + x_3\frac{\partial}{\partial x_4}\right)$$

thus $\bar{\nabla}_{e_2} e_1 = \bar{\nabla}_{\vec{e}_2}\vec{e}_1 = 0 + 0 = 0$

$$\bar{\nabla}_{e_1} e_2 = \bar{\nabla}_{\vec{e}_1}\vec{e}_2 = 0 + 0 = 0$$

$$\bar{\nabla}_{e_2} e_2 = \bar{\nabla}_{\vec{e}_2}\vec{e}_2 = -(\sqrt{2})^2 x_4 \frac{\partial x_3}{\partial x_3}\frac{\partial}{\partial x_4} - (\sqrt{2})^2 x_3 \frac{\partial x_4}{\partial x_4}\frac{\partial}{\partial x_3} = -2\left(x_3\frac{\partial}{\partial x_3} + x_4\frac{\partial}{\partial x_4}\right)$$

By extending n_1, n_2 to vector fields on $\mathbb{R}^4$ given by

$$\overline{n_1} : \mathbb{R}^4 \ni (x_1, x_2, x_3, x_4) \longmapsto (x_1, x_2, x_3, x_4) = \left(x_1\frac{\partial}{\partial x_1} + x_2\frac{\partial}{\partial x_2} + x_3\frac{\partial}{\partial x_3} + x_4\frac{\partial}{\partial x_4}\right)$$

$$\overline{n_2} : \mathbb{R}^4 \ni (x_1, x_2, x_3, x_4) \longmapsto (-x_1, -x_2, x_3, x_4) = \left(-x_1\frac{\partial}{\partial x_1} - x_2\frac{\partial}{\partial x_2} + x_3\frac{\partial}{\partial x_3} + x_4\frac{\partial}{\partial x_4}\right)$$

we can use the fact that $\langle S_{n_k}(e_i), e_j\rangle = -\langle \bar{\nabla}_{e_i} n_k, e_j\rangle = \langle \bar{\nabla}_{e_i} e_j, n_k\rangle$ to conclude that

$$\langle S_{n_1}(e_1), e_1\rangle = \langle \bar{\nabla}_{e_1} e_1, n_1\rangle = -2(x_1^2 + x_2^2)\Big|_{\substack{x_1 = \frac{1}{\sqrt{2}}\cos\theta \\ x_2 = \frac{1}{\sqrt{2}}\sin\theta}} = -2\cdot\frac{1}{2} = -1$$

$$\langle S_{n_1}(e_1), e_2\rangle = \langle \bar{\nabla}_{e_1} e_2, n_1\rangle = 0$$

$$\langle S_{n_1}(e_2), e_1\rangle = \langle \bar{\nabla}_{e_2} e_1, n_1\rangle = 0$$

$$\langle S_{n_1}(e_2), e_2\rangle = \langle \bar{\nabla}_{e_2} e_2, n_1\rangle = -2(x_3^2 + x_4^2)\Big|_{\substack{x_3 = \frac{1}{\sqrt{2}}\cos\varphi \\ x_4 = \frac{1}{\sqrt{2}}\sin\varphi}} = -2\cdot\frac{1}{2} = -1$$

thus $S_{n_1} = \begin{pmatrix} -1 & 0 \\ 0 & -1 \end{pmatrix}$; and

$$\langle S_{n_2}(e_1), e_1\rangle = \langle \bar{\nabla}_{e_1} e_1, n_2\rangle = 2(x_1^2 + x_2^2)\Big|_{\substack{x_1 = \frac{1}{\sqrt{2}}\cos\theta \\ x_2 = \frac{1}{\sqrt{2}}\sin\theta}} = 2\cdot\frac{1}{2} = 1$$

$$\langle S_{n_2}(e_1), e_2\rangle = \langle \bar{\nabla}_{e_1} e_2, n_2\rangle = 0$$

$$\langle S_{n_2}(e_2), e_1\rangle = \langle \bar{\nabla}_{e_2} e_1, n_2\rangle = 0$$

$$\langle S_{n_2}(e_2), e_2\rangle = \langle \bar{\nabla}_{e_2} e_2, n_2\rangle = -2(x_3^2 + x_4^2)\Big|_{\substack{x_3 = \frac{1}{\sqrt{2}}\cos\varphi \\ x_4 = \frac{1}{\sqrt{2}}\sin\varphi}} = -2\cdot\frac{1}{2} = -1$$

thus $S_{n_2} = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$

c) When we view $\vec{x}$ as an immersion from T^2 to $S^3(1)$, the normal space is of dimension 1 spanned by n_2 only, since n_1 is a normal vector of $S^3(1)$ in $\mathbb{R}^4$. Thus for each $p \in T^2$ we have

$$H(p) = \frac{1}{2}(\operatorname{trace} S_{n_2}) \cdot n_2 = \frac{1}{2} \cdot 0 \cdot n_2 = 0$$

which means that $\vec{x}: T^2 \hookrightarrow S^3(1)$ is a minimal immersion.

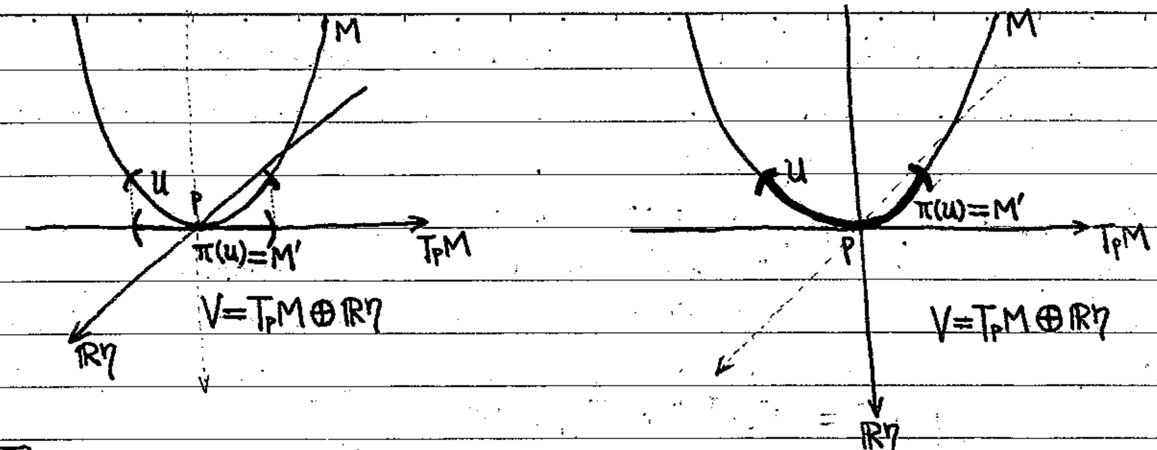
[Another way to see c): let $\nabla, \tilde{\nabla}, \bar{\nabla}$ be the Riemannian connections of T^2, $S^3(1)$, $\mathbb{R}^4$ respectively. Then the second fundamental form of T^2 as a submanifold of $S^3(1)$ is $B(X,Y) = (\tilde{\nabla}_X Y)^\perp = ((\bar{\nabla}_X Y)^T)^\perp = (\bar{\nabla}_X Y - \langle \bar{\nabla}_X Y, x\rangle x)^\perp$. Simple computation yields $B(e_1,e_1) = \bar{\nabla}_{e_1} e_1 + x$, $B(e_2,e_2) = \bar{\nabla}_{e_2} e_2 + x$. Thus $H = \frac{1}{2}(B(e_1,e_1) + B(e_2,e_2)) = 0$, where we used the observation that $H = \operatorname{trace} B$. Hence T^2 is a minimal submanifold of $S^3(1)$.]

Remark: Lawson's conjecture states that the Clifford torus is the only minimally embedded torus in the three-sphere S^3. In March 2012, just a few months before this solution was written, Simon Brendle proved this conjecture ("Embedded Minimal Tori in S^3 and the Lawson Conjecture", http://arxiv.org/abs/1203.6597).

9. Let $f: M^n \to \mathbb{R}^{m+n}$ be an immersion. Let $\eta \in (T_pM)^\perp$, $p \in M$ and $V = T_pM \oplus \mathbb{R}\eta \subset \mathbb{R}^{m+n}$, where $\mathbb{R}\eta = \{\lambda\eta \mid \lambda \in \mathbb{R}\}$. Let $\pi: \mathbb{R}^{m+n} \to \mathbb{R}^{m+n}$ be the orthogonal projection onto $T_pM \oplus \mathbb{R}\eta$. Since η is transversal to M at p, $\pi|\mathcal{U}$ is an embedding, where $\mathcal{U}$ is a sufficiently small neighborhood of p in M. Let $M' = \pi(\mathcal{U}) \subset T_pM \oplus \mathbb{R}\eta$ and let $S'_\eta: T_pM' = T_pM \to T_pM'$ be the operator associated to the second fundamental form of M' at p in the direction of η. Show that $S'_\eta = S_\eta$, where $S_\eta: T_pM \to T_pM$ is the operator associated to the second fundamental form of M at p in the direction of η.

Hint: Let N and N' be normal fields along $\mathcal{U}$ and $\pi(\mathcal{U})$, respectively, such that $N(p) = N'(p) = \eta$. Then: if $X \in T_pM$, $S_\eta(X) = -(\bar{\nabla}_X N)^T$ and $S'_\eta(X) = -(\bar{\nabla}_X N')^T$. Show that it is possible to choose N in such a way that $N' = d\pi(N)$ and observe that the restriction $d\pi|T_pM = id$. Hence, at p, $S'_\eta(X) = -(\bar{\nabla}_X(d\pi N))^T = -d\pi(\bar{\nabla}_X N)^T = S_\eta(X)$.

$\boxed{0^\circ}$ Two pictures in mind:



$\boxed{1^\circ}$ Why is $\pi|\mathcal{U} : M \supset \mathcal{U} \longrightarrow T_pM \oplus \mathbb{R}\eta$ an embedding? First we show that $\pi\circ f : M \to df(T_pM) \oplus \mathbb{R}\eta$ is an immersion at $p \in M$, then we apply Proposition 3.7 of Chapter 0 (P.13). To see that $\pi\circ f : M \longrightarrow T_pM \oplus \mathbb{R}\eta$ is an immersion, first note that $d\pi = \pi$ since π is linear, then recall that $df : T_pM \longrightarrow df(T_pM) \subset \mathbb{R}^{m+n}$ is an injection since $f: M^n \longrightarrow \mathbb{R}^{m+n}$ is an immersion. Also note that $d\pi : df(T_pM) \to df(T_pM) \oplus \mathbb{R}\eta$ is an injection, since $d\pi = \pi : \mathbb{R}^{m+n} \longrightarrow df(T_pM) \oplus \mathbb{R}\eta$ is an orthogonal projection, and thus $d\pi : df(T_pM) \longrightarrow df(T_pM) \oplus \mathbb{R}\eta$ is the natural identification $v \longmapsto (v, 0)$ (since by assumption η is transversal to $df(T_pM)$). Hence the composition $d(\pi\circ f) = d\pi \circ df : T_pM \longrightarrow df(T_pM) \oplus \mathbb{R}\eta$ is also an injection, which proves that $\pi\circ f : M \longrightarrow df(T_pM) \oplus \mathbb{R}\eta$ is an immersion. Now by citing Proposition 3.7 in Chapter 0 we are sure of the existence of the sufficiently small neighborhood $\mathcal{U}$ of $p \in M$. In the follows we are going to omit the immersion f in our solution so as to simplify the notations, that is, we use π to denote $\pi\circ f$, T_pM to denote $df(T_pM)$, etc.

$\boxed{2^\circ}$ Let $M' = \pi(\mathcal{U}) \subset T_pM \oplus \mathbb{R}\eta$, then T_pM' can be canonically identified with T_pM since $\pi|\mathcal{U} : \mathcal{U} \longrightarrow T_pM \oplus \mathbb{R}\eta$ is an embedding, which implies that $d\pi_p : T_p\mathcal{U} = T_pM \longrightarrow T_{\pi(p)}(\pi(\mathcal{U}))$ is an isomorphism between vector spaces. (Here $d\pi_p$ is actually $d(\pi\circ f)_p$, of course.)

$\boxed{3^\circ}$ Let N' be a normal field along $\pi(\mathcal{U})$, with $N'(p) = \eta \in (T_pM)^\perp$. We construct a normal field N along $\mathcal{U}$ such that $d\pi(N) = N'$ and $N(p) = N'(p) = \eta$.

- First note that since $\pi|_{\mathcal{U}} : \mathcal{U} \longrightarrow \pi(\mathcal{U})$ is a diffeomorphism (since it is an embedding), we can define $N \in \mathfrak{X}(\mathcal{U})$ by

$$N(f)(q) = \left[N'\big(\underbrace{f\circ(\pi|_{\mathcal{U}})^{-1}}_{\in\, C^\infty(M') = C^\infty(\pi(\mathcal{U}))}\big)\right]\big((\pi|_{\mathcal{U}})(q)\big), \quad \forall f \in C^\infty(\mathcal{U}),\ \forall q \in \mathcal{U}.$$

• Then we show that $N' = d\pi(N)$, i.e. N' is π-related to N, or equivalently that N' is the push-forward of the vector field N viewed as a vector field on $\mathbb{R}^{m+n}$. For any $f \in C^\infty(\pi(\mathcal{U}))$, $f \circ (\pi|_\mathcal{U}) \in C^\infty(\mathcal{U})$, and we have

$$\big(d\pi(N)\big)(f) = N\big(f\circ(\pi|_\mathcal{U})\big)\circ(\pi|_\mathcal{U})^{-1} = \Big(\big[N'\big(f\circ(\pi|_\mathcal{U})\circ(\pi|_\mathcal{U})^{-1}\big)\big]\circ(\pi|_\mathcal{U})\Big)\circ(\pi|_\mathcal{U})^{-1}$$
$$= N'(f) \in C^\infty(\pi(\mathcal{U})).$$

where we used the definition of the push-forward of the vector field N (cf. John Lee, "Introduction to Smooth Manifold", Proposition 3.18 in Chapter 3) and the definition of $N \in \mathfrak{X}(\mathcal{U})$. Let us point out that when viewed as a vector field on $\mathbb{R}^{m+n}$ we should define N as derivations on $C^\infty(\mathbb{R}^{m+n})$ instead of $C^\infty(\mathcal{U})$, but we can always extend a function in $C^\infty(\mathcal{U})$ to a function in $C^\infty(\mathbb{R}^{m+n})$, and $N'(f\circ(\pi|_\mathcal{U})^{-1})$ is independent of the extension chosen since $N' \in \mathfrak{X}(\pi(\mathcal{U}))$ (this is another example of an argument like "the integral curves of N' remains in $\pi(\mathcal{U})$ since $N' \in \mathfrak{X}(\pi(\mathcal{U}))$" etc.)

• Next we claim that $N \in \mathfrak{X}(\mathcal{U})^\perp$. In fact, at any point $q \in \mathcal{U}$, let $X(q)$ be a tangent vector at q to $\mathcal{U}$, we have

$$\langle N(q), X(q)\rangle_q \overset{①}{=\!=} \langle (d\pi_q)^{-1}(N'(\pi(q))), X(q)\rangle_q \overset{②}{=\!=} \langle (d\pi_q)^*(N'(\pi(q))), X(q)\rangle_q$$

$$\overset{③}{=\!=} \langle N'(\pi(q)), d\pi_q(X(q))\rangle_{\pi(q)} \overset{④}{=\!=} 0$$

where $\langle\,,\rangle$ denotes the canonical Euclidean metric on $\mathbb{R}^{m+n}$ (being the ambient space for both $\mathcal{U}$ and $\pi(\mathcal{U}) = M'$); at ① we used the fact that if N is a push-forward of the vector field N' then $N'(q)$ push-forwards to $N(\pi(q))$ under $d\pi_q$, and the fact that $\pi|_\mathcal{U}: \mathcal{U} \to \pi(\mathcal{U})$ is a diffeomorphism; at ② we used the assumption that π is an orthogonal projection, and that π is linear, to deduce that $(d\pi_q)^{-1} = (\pi)^{-1} = \pi^* = (d\pi_q)^*$ (recall that an orthogonal projection P on a Hilbert space $\mathcal{H}$ is defined as a bounded linear map satisfying $P^2 = P$ and $P = P^*$, and $\pi|_{T_pM\oplus\mathbb{R}\eta} = Id$, hence on $T_pM\oplus\mathbb{R}\eta$ one has $\pi^*\pi = \pi\pi^* = \pi^2 = \pi = Id \Rightarrow \pi^* = \pi^{-1}$); at ③ we used the definition of the adjoint of a bounded linear map; (watch for change of the sub-index for the inner product bracket) at ④ we used the observation that $d\pi_q(X(q)) \in T_pM' = T_p(\pi(\mathcal{U}))$ and $N'(\pi(q)) \in \mathfrak{X}(M')^\perp = \mathfrak{X}(\pi(\mathcal{U}))^\perp$. Here again we point out that $(d\pi_q)^{-1}(N'(\pi(q)))$ is a vector in $T_pM\oplus\mathbb{R}\eta \subset T_q\mathbb{R}^{m+n}$, not a tangent vector in T_qM, since it is not tangent to M.

No.

Date

• Finally we conclude that we constructed $N \in \mathfrak{X}(\mathcal{U})^{\perp}$ satisfying $d\pi(N) = N'$.

$\boxed{4°}$ Let $\bar{\nabla}$ denote for the Levi-Civita connection on $\mathbb{R}^{m+n}$, and let X be a tangent vector in T_pM. For any $Y \in T_pM$, one has

$$B^{M'}(X,Y) = \bar{\nabla}_{\bar{X}}\bar{Y} - (\bar{\nabla}_{\bar{X}}\bar{Y})^{\text{tangent}} = B^M(X,Y)$$

since $T_pM = T_pM'$ and thus $(\bar{\nabla}_{\bar{X}}\bar{Y})^{\text{tangent}}$ for M and M' coincide. Also note that since $N'(p) = N(p) = \eta$ one has

$$\langle B^{M'}(X,Y), N' \rangle(p) = \langle B^M(X,Y), N \rangle(p)$$

(better alternative: $\langle -(\bar{\nabla}_X(d\pi(N)))^T, Y\rangle(p) = \langle -\bar{\nabla}_X(d\pi(N)), Y\rangle(p) = \langle d\pi(-\bar{\nabla}_X N), Y\rangle(p)$ $= \langle [d\pi(-\bar{\nabla}_X N)]^T, Y\rangle(p) = \langle -d\pi[(\bar{\nabla}_X N)^T], Y\rangle(p)$)

which gives

$$\langle -(\bar{\nabla}_X(d\pi(N)))^T, Y\rangle(p) = \langle -(\bar{\nabla}_X N')^T, Y\rangle(p) = \langle B^{M'}(X,Y), N'\rangle(p) = \langle B^M(X,Y), N\rangle(p) =$$
$$= \langle -(\bar{\nabla}_X N)^T, Y\rangle(p) \overset{(*)}{=\!=} \langle -d\pi[(\bar{\nabla}_X N)^T], Y\rangle(p)$$

where at $(*)$ we used the hypothesis that $d\pi|_{T_pM} : T_pM \to T_pM$ is the identity map and the fact that $(\bar{\nabla}_X N)^T \in T_pM$. It follows that $-(\bar{\nabla}_X(d\pi(N)))^T = -d\pi[(\bar{\nabla}_X N)^T]$ at p, and that $S'_\eta(X) = -(\bar{\nabla}_X(d\pi(N)))^T = -d\pi[(\bar{\nabla}_X N)^T] = -(\bar{\nabla}_X N)^T = S_\eta(X)$ at p.

Remark 1 In 4°, $(*)$ seems unnecessary. We did that because the hint needs $-(\bar{\nabla}_X(d\pi(N)))^T = -d\pi[(\bar{\nabla}_X N)]^T$ at p, which follows from the linearity of $d\pi$ and thus its commutativity with $\bar{\nabla}$ and $(\cdot)^T$.

Remark 2 The problem says: "Since η is transversal to M at p, $\pi|\mathcal{U}$ is an embedding, where ..." However, even if η is not transversal to M at p, $\pi|\mathcal{U}$ is still an embedding; since in this case $T_pM \oplus \mathbb{R}\eta = T_pM$, and $d\pi_p : T_pM \to T_pM$ is exactly the identity map, thus π is an immersion of M to T_pM at p, which further gives the locally embedding property of π by Proposition 3.7 of Chapter 0. This still makes sense of the problem but changed the vivid geometric picture in this problem (see remarks below).

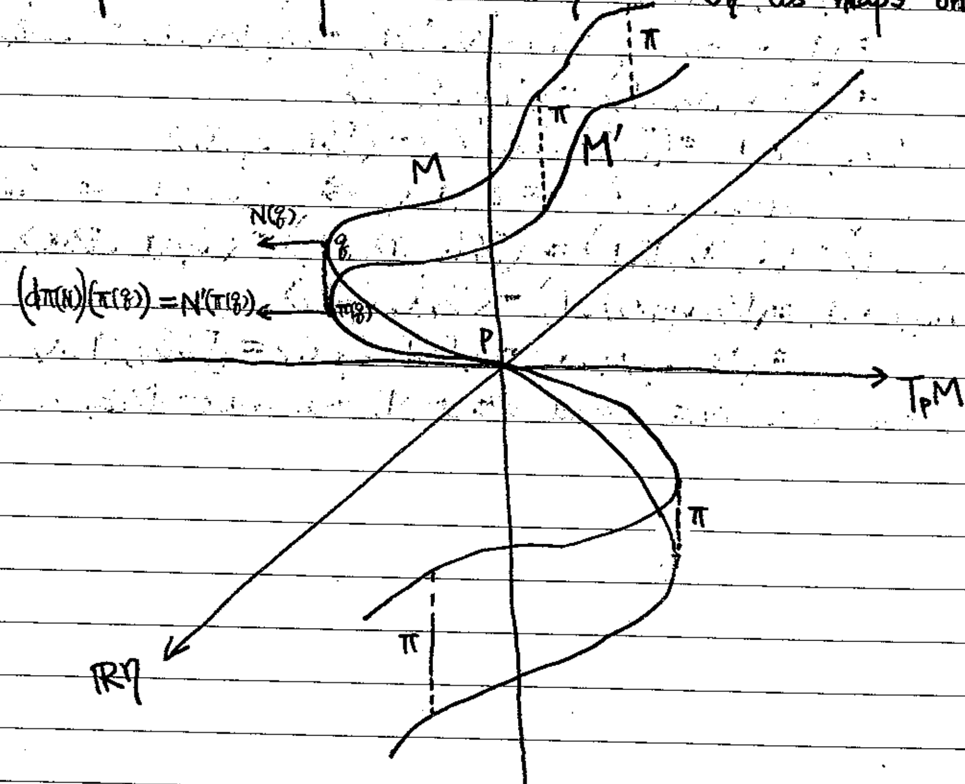
Remark 3 Why do we need to construct $N' = d\pi(N)$ on a neighborhood of p instead of picking N', N arbitrarily, only satisfying $N'(p) = N(p) = \eta$? The reason is that in the latter case we can not pull $d\pi$ out of the covariant differentiation $\bar{\nabla}$ and obtain $-\bar{\nabla}_X(d\pi(N)) = -d\pi(\bar{\nabla}_X N)$, since $\bar{\nabla}$ need information of N, N' in a neighborhood of p (being a local operator only, not a tensor).

Remark 4 The geometric interpretation of this problem, as suggested by

Prof. Mark Stern, is "approximating a given manifold by a better controlled manifold at one single point while preserving the operator $S_\eta(\cdot)$". Another simpler approximation of similar flavor is as follows: given a (1-dimensional) curve in $\mathbb{R}^3$, assume it has curvature $1/r$ at p with $r > 0$, one can use a circle, tangent to the curve at p with radius r, to approximate the curve at p, up to the precision of $o(x^2)$ (the curvature corresponds to the quadratic error term).

<u>Remark 5</u> A confusion may arise if one is being pedantic, since all the technical details about the immersion of M into $\mathbb{R}^{m+n}$ is black-boxed in the immersion $f: M \to \mathbb{R}^{m+n}$ and is encoded in the simplified notation (π for $\pi \circ f$, etc.).

<u>Remark 6</u> A better and more generic picture (than those shown in 0°, which corresponds to the special case $S'_\eta = 0 = S_\eta$ as maps on T_pM):



where the blue curve is a "space curve" in the ambient space, and the red curve (being the projection of the blue curve) is a "plane curve" in the linear subspace $T_pM \oplus \mathbb{R}\eta$.

No.

Date

10. Let $f: M^n \longrightarrow \overline{M}^{n+k}$ be an isometric immersion and let $S_\eta: TM \longrightarrow TM$ be the operator associated to the second fundamental form of f along the normal field η. Consider S_η as a tensor of order 2 given by $S_\eta(X,Y) = \langle S_\eta(X), Y\rangle$, $X, Y \in \mathcal{X}(M)$. Observe that saying the operator S_η is self-adjoint is equivalent to saying that the tensor S_η is symmetric, that is, $S_\eta(X,Y) = S_\eta(Y,X)$. Prove that for all $V \in \mathcal{X}(M)$, the tensor $\nabla_V S_\eta$ is symmetric.

Hint: Differentiating $\langle S_\eta(X), Y\rangle = \langle X, S_\eta(Y)\rangle$ with respect to V, we obtain $\langle \nabla_V(S_\eta(X)), Y\rangle + \langle S_\eta(X), \nabla_V Y\rangle = \langle \nabla_V X, S_\eta Y\rangle + \langle X, \nabla_V(S_\eta Y)\rangle$. Using the fact that $\langle (\nabla_V S_\eta)X, Y\rangle = \langle \nabla_V(S_\eta X), Y\rangle - \langle S_\eta(\nabla_V X), Y\rangle$ and the previous expression, we obtain easily that $\langle (\nabla_V S_\eta)X, Y\rangle = \langle X, (\nabla_V S_\eta)Y\rangle$.

- Proof: By definition and the self-adjoint property of S_η one has for any $X, Y \in \mathcal{X}(M)$ and $V \in \mathcal{X}(M)$

$$(\nabla_V S_\eta)(X,Y) = (\nabla S_\eta)(X,Y,V) = V\langle S_\eta(X), Y\rangle - \langle S_\eta(\nabla_V X), Y\rangle - \langle S_\eta(X), \nabla_V Y\rangle$$
$$= \langle \nabla_V(S_\eta(X)), Y\rangle + \cancel{\langle S_\eta(X), \nabla_V Y\rangle} - \langle S_\eta(\nabla_V X), Y\rangle - \cancel{\langle S_\eta(X), \nabla_V Y\rangle}$$
$$= \langle \nabla_V(S_\eta(X)), Y\rangle - \langle \nabla_V X, S_\eta(Y)\rangle$$

Similarly, $(\nabla_V S_\eta)(Y,X) = \langle \nabla_V(S_\eta(Y)), X\rangle - \langle \nabla_V Y, S_\eta(X)\rangle$.

Since S_η is symmetric, $S_\eta(X,Y) = S_\eta(Y,X) \Rightarrow V(S_\eta(X,Y)) = V(S_\eta(Y,X))$

$\Rightarrow \langle \nabla_V(S_\eta(X)), Y\rangle + \langle S_\eta(X), \nabla_V Y\rangle = \langle \nabla_V(S_\eta(Y)), Y\rangle + \langle S_\eta(Y), \nabla_V X\rangle$.

Hence
$$(\nabla_V S_\eta)(X,Y) = \langle \nabla_V(S_\eta(X)), Y\rangle - \langle \nabla_V X, S_\eta(Y)\rangle$$
$$= \langle \nabla_V(S_\eta(Y)), X\rangle - \langle \nabla_V Y, S_\eta(X)\rangle = (\nabla_V S_\eta)(Y,X).$$

In words, this shows that for all $V \in \mathcal{X}(M)$ the tensor $\nabla_V S_\eta$ is symmetric.

11. Let $f: \overline{M}^{n+1} \longrightarrow \mathbb{R}$ be a differentiable function. Define the Hessian, Hess f of f at $p \in \overline{M}$ as the linear operator
$\text{Hess} f: T_p\overline{M} \longrightarrow T_p\overline{M}$, $(\text{Hess} f)Y = \overline{\nabla}_Y \text{grad} f$, $Y \in T_p\overline{M}$, where $\overline{\nabla}$ is the Riemannian connection of $\overline{M}$. Let a be a regular value of f and let $M^n \subset \overline{M}^{n+1}$ be the hypersurface in $\overline{M}$ defined by $M = \{p \in \overline{M} ; f(p) = a\}$. Prove that:
a) The Laplacian $\overline{\Delta} f$ is given by $\overline{\Delta} f = \text{trace Hess} f$.
b) If $X, Y \in \mathfrak{X}(\overline{M})$, then $\langle (\text{Hess} f) Y, X \rangle = \langle Y, (\text{Hess} f) X \rangle$. Conclude that Hess f is self-adjoint, hence determines a symmetric bilinear form on $T_p\overline{M}$, $p \in \overline{M}$, given by $(\text{Hess} f)(X, Y) = \langle (\text{Hess} f) X, Y \rangle$, $X, Y \in T_p\overline{M}$.
c) The mean curvature H of $M \subset \overline{M}$ is given by $nH = -\text{div}\left(\frac{\text{grad} f}{\|\text{grad} f\|}\right)$.
Hint: Take an orthonormal frame $E_1, \cdots, E_n$, $E_{n+1} = \frac{\text{grad} f}{\|\text{grad} f\|} = \eta$ in a neighborhood of $p \in M$ in $\overline{M}$ and use the definition of divergence in Exercise 8, Chapter 3, to obtain

$$nH = \text{trace } S_\eta = \sum_{i=1}^{n} \langle S_\eta(E_i), E_i \rangle = -\sum_{i=1}^{n} \langle \overline{\nabla}_{E_i} \eta, E_i \rangle - \langle \overline{\nabla}_\eta \eta, \eta \rangle = \sum_{i=1}^{n+1} \langle \overline{\nabla}_{E_i} \eta, E_i \rangle$$
$$= -\text{div}_{\overline{M}} \eta = -\text{div}\left(\frac{\text{grad} f}{\|\text{grad} f\|}\right).$$

d) Observe that every embedded hypersurface $M^n \subset \overline{M}^{n+1}$ is locally the inverse image of a regular value. Conclude from (c) that the mean curvature H of such a hypersurface is given by $H = -\frac{1}{n} \text{div} N$, where N is an appropriate local extension of the unit normal vector field on $M^n \subset \overline{M}^{n+1}$.

- Proof: $\boxed{a)}$ By the definition we have seen on P83 Exercise 8 of Chapter 3, for all $p \in \overline{M}$ we have

$$\overline{\Delta} f(p) = \text{div}(\text{grad} f)(p) = \text{trace of the linear mapping } Y(p) \longmapsto (\overline{\nabla}_Y \text{grad} f)(p)$$
$$= \text{trace of the linear mapping } Y(p) \longmapsto ((\text{Hess} f) Y)(p)$$
$$= (\text{trace Hess} f)(p).$$

By the arbitrariness of $p \in \overline{M}$, this gives. $\overline{\Delta} f = \text{trace Hess} f$ on $\overline{M}$.
$\boxed{b)}$ Since $\overline{\nabla}$ is torsion-free, for all $X, Y \in \mathfrak{X}(M)$ one has

$$YX - XY = [Y, X] = \overline{\nabla}_Y X - \overline{\nabla}_X Y \Rightarrow YX - \overline{\nabla}_Y X = XY - \overline{\nabla}_X Y$$

For any $X \in \mathfrak{X}(M)$, let $\varphi_t(p)$ be the flow of X starting from p, i.e.

$$\begin{cases} \frac{d}{dt} \varphi_t(p) = X(\varphi_t(p)) & t \geq 0 \\ \varphi_0(p) = p \end{cases}$$

Then $df_p(X) = \frac{d}{dt}\Big|_{t=0} f \circ (\varphi_t(p)) = (Xf)(p)$.

No.

Date

Hence $\langle (\mathrm{Hess} f)Y, X\rangle(p) = \langle \bar\nabla_Y \operatorname{grad} f, X\rangle(p) = Y\langle \operatorname{grad} f, X\rangle(p) - \langle \operatorname{grad} f, \bar\nabla_Y X\rangle(p)$

$$= Y\big(df_p(X(p))\big) - df_p\big((\bar\nabla_Y X)(p)\big)$$
$$= YXf(p) - [(\bar\nabla_Y X)f](p)$$
$$= XYf(p) - [(\bar\nabla_X Y)f](p)$$
$$= X\big(df_p(Y(p))\big) - df_p\big((\bar\nabla_X Y)(p)\big)$$
$$= X\langle \operatorname{grad} f, Y\rangle(p) - \langle \operatorname{grad} f, \bar\nabla_Y X\rangle(p)$$
$$= \langle \bar\nabla_X \operatorname{grad} f, Y\rangle(p) = \langle (\mathrm{Hess} f)X, Y\rangle(p) = \langle Y, (\mathrm{Hess} f)X\rangle(p)$$

for all $p\in\bar M$. The arbitrariness of $p\in\bar M$ gives

$$\langle (\mathrm{Hess} f)Y, X\rangle = \langle Y, (\mathrm{Hess} f)X\rangle = \langle (\mathrm{Hess} f)^* Y, X\rangle, \ \forall X, Y\in\mathfrak{X}(M).$$

Hence $\mathrm{Hess} f = (\mathrm{Hess} f)^*$, i.e. $\mathrm{Hess} f$ is self-adjoint. By standard functional analysis one knows that

$$(\mathrm{Hess} f)^\flat(X, Y) = \langle (\mathrm{Hess} f)X, Y\rangle, \quad X, Y\in T_p\bar M, \ p\in\bar M$$

is indeed a symmetric bilinear form on $T_p\bar M$, $p\in\bar M$.

$\boxed{c)}$ Span $\dfrac{\operatorname{grad} f}{|\operatorname{grad} f|}$ into an orthonormal frame in a neighborhood of $p\in M$ in $\bar M$. Note here the frame is orthonormal but not necessarily geodesic. Note also that we can find an orthonormal frame $E_1, \dots, E_n$, $E_{n+1} = \frac{\operatorname{grad} f}{|\operatorname{grad} f|} = \eta$ because the real number a in the definition $M = \{p\in\bar M;\ f(p) = a\}$ is a regular value of f. Indeed, since a is a regular value of f, $df_p: T_p\bar M \to T_a\mathbb{R} = \mathbb{R}$ is surjective for any $p\in M$, and thus $\operatorname{grad} f(p)$ is non-zero for all $p\in M$ (here we used the definition of $\operatorname{grad} f$: $\langle \operatorname{grad} f(p), v\rangle = df_p(v)$ for all $v\in T_p\bar M$ and the Riesz Representation Theorem for real linear functionals on $T_p\bar M$ to conclude directly that $df_p = \operatorname{grad} f(p)$, but it also suffices even if we don't refer to this important theorem in functional analysis). It follows that $\eta = \dfrac{\operatorname{grad} f}{|\operatorname{grad} f|}$ is well-defined and regular at $p\in M\subseteq\bar M$, and we can choose a submanifold S of $\bar M$, passing through p, normal to η at p, with dimension $\dim S = \dim\bar M - 1 = n-1$, such that p has a neighborhood diffeomorphic to $S\times\varphi_p^\eta(-\varepsilon,\varepsilon)$ for some $\varepsilon>0$, where $\varphi_p^\eta(t)$ denotes for the flow associated to η passing through p at $t=0$. (This is essentially established as "Canonical Form for a Regular Vector Field", cf. John Lee "Introduction to

Smooth Manifolds", Theorem 13.5, pp. 331. We proved this theorem manually in our solution to Exercise 5 e) of Chapter 3.) It is now clear how to choose the desired orthonormal frame around p: first choose an orthonormal frame in S, then push them forward to smooth vector fields on a neighborhood of p, next add η to this collection, and finally use a Gram-Schmidt orthonormalization process to obtain $E_1, \cdots, E_n, E_{n+1} = \frac{\operatorname{grad} f}{|\operatorname{grad} f|} = \eta$. Hence

$$\begin{aligned}
nH(p) &= (\operatorname{trace} S_\eta) \text{ at } p \\
&= \sum_{i=1}^{n} \langle S_\eta(E_i), E_i \rangle(p) = -\sum_{i=1}^{n} \langle \bar{\nabla}_{E_i}\eta, E_i \rangle(p) \\
&= -\sum_{i=1}^{n} \langle \bar{\nabla}_{E_i}\eta, E_i \rangle(p) - \langle \bar{\nabla}_\eta \eta, \eta \rangle(p) = -\sum_{i=1}^{n} \langle \bar{\nabla}_{E_i}\eta, E_i \rangle(p) \\
&= -\left(\text{trace of the linear map } X(p) \longmapsto (\bar{\nabla}_X \eta)(p)\right) \\
&= -(\operatorname{div}_{\bar{M}} \eta)(p) = -\operatorname{div}_{\bar{M}}\left(\frac{\operatorname{grad} f}{|\operatorname{grad} f|}\right)(p) \qquad \forall\, p \in M
\end{aligned}$$

where we used the observation that $\langle \bar{\nabla}_\eta \eta, \eta \rangle \equiv 0$ in a neighborhood around p since $\langle \eta, \eta \rangle \equiv 1$ there. We also observed that $E_1(p), \cdots, E_n(p)$ is an orthonormal basis of $T_pM \subset T_p\bar{M}$ since $f|_M \equiv a$ implies $\langle \operatorname{grad} f(p), w \rangle = df_p(w) = 0$ for all $w \in T_pM$ and thus $\operatorname{grad} f(p) \in (T_pM)^\perp$. But $\dim T_pM = n$, thus $(T_pM)^\perp = \operatorname{span}\{\operatorname{grad} f(p)\}$ and hence $T_pM = \operatorname{span}\{E_1(p), \cdots, E_n(p)\}$. Finally, by the arbitrariness of $p \in M$, we have established

$$nH = -\operatorname{div}_{\bar{M}}\left(\frac{\operatorname{grad} f}{|\operatorname{grad} f|}\right) \quad \text{on } M.$$

$\boxed{d)}$ Every embedded hypersurface $M^n \subset \bar{M}^{n+1}$ is diffeomorphic to its image in $\bar{M}^{n+1}$, i.e. the immersion $f: M^n \to \bar{M}^{n+1}$ has rank n. By the Rank Theorem (cf. John Lee "Introduction to Smooth Manifolds", Theorem 5.13, pp. 107) for any $p \in \bar{M}$ there exists a local coordinate system $(\mathcal{U}, x)$ of $\bar{M}$ around p such that $x(p) = 0$ and $\mathcal{U} \cap M = \{q \in \mathcal{U} : x^{n+1}(q) = 0\}$. Define a smooth function F in $\mathcal{U}$ by setting $F(q) = x^{n+1}(q)$, then 0 is a regular value of F since $dF_q : T_q\bar{M} \to T_{F(q)}\mathbb{R}$ has constant rank 1 at each $q \in \mathcal{U}$ and thus every $a \in \mathbb{R}$ is a regular value of F (including trivially those $a \notin F(\mathcal{U})$ by definition, cf. P.17 Example 4.3 of Chapter 0). We know by this construction that $F^{-1}(0) = \mathcal{U} \cap M$. Let η be a unit normal vector field on $M^n \subset \overline{M}^{n+1}$. We know from our solution to c) that $N = \frac{\operatorname{grad} F}{|\operatorname{grad} F|}$ is also a unit normal vector field when restricted to M. Since M^n has codimension 1 in $\overline{M}^{n+1}$, N and η coincide as long as we choose η according to appropriate local orientation around p. It follows from c) then that $H(p) = -\frac{1}{n} \operatorname{div}\left(\frac{\operatorname{grad} F}{|\operatorname{grad} F|}\right)(p)$ $= -\frac{1}{n} \operatorname{div} N(p)$. By the arbitrariness of $p \in M$, this implies

$$H = -\frac{1}{n} \operatorname{div} N \text{ on } M.$$

Remark: Although there are two equally nice choices of η, in c) we are actually imposing $\eta = \frac{\operatorname{grad} f}{|\operatorname{grad} f|}$ to avoid the possible confusion.

12. (Singularities of a Killing field). Let X be a Killing vector field on a Riemannian manifold M. Let $N=\{p\in M;\ X(p)=0\}$. Prove that:

a) If $p\in N$, and $V\subset M$ is a normal neighborhood of p, with $q\in N\cap V$, then the radial geodesic segment γ joining p to q is contained in N. Conclude that $\gamma\cap V\subset N$.

b) If $p\in N$, there exists a neighborhood $V\subset M$ of p such that $V\cap N$ is a submanifold of M (this implies that every connected component of N is a submanifold of M).

Hint: Proceed by induction, using (a). If p is isolated, nothing has to be done. In the contrary case, let $V\subset M$ be a normal neighborhood of p such that there exists $q_1\in V\cap N$ and consider the radial geodesic γ_1 joining p to q_1. If $V\cap N=\gamma_1$, by (a), the proof is complete. Otherwise, let $q_2\in V\cap N-\{\gamma_1\}$ and let γ_2 be the radial geodesic joining p to q_2. Let $Q\subset T_pM$ be the subspace generated by the vectors $\exp_p^{-1}(q_1)$ and $\exp_p^{-1}(q_2)$ and let $N_2=\exp_p(Q\cap\exp_p^{-1}(V))$. Show that for all $t\in\mathbb{R}$, the restriction of the differential $(dX_t)_p$ of the flow $X_t: M\longrightarrow M$, to Q, is the identity; Conclude now that $N_2\subset V\cap N$. Proceed in this way until the dimension of T_pM is exhausted.

c) The condimension, as a submanifold of M, of a connected component N_k of N is even. Assume the following fact: if a sphere has a non-vanishing differentiable vector field on it then its dimension must be odd (for a proof, see Armstrong, [Ar], P.198).

Hint: Let $E_p=(T_pN_k)^{\perp}$ and let $V\subset M$ be a normal neighborhood of p. Set $N_k^{\perp}=\exp_p(E_p\cap\exp_p^{-1}(V))$. Since, for all t, $(dX_t)_p: E_p\to E_p$, we have that X is tangent to $N_k^{\perp}$. On the other hand, $X\neq 0$ is tangent to the geodesic spheres of $N_k^{\perp}$ with center p. From the theorem mentioned above, the dimension of such a sphere is odd. Hence $\dim N_k^{\perp}=\dim E$ is even.

• Proof: $\boxed{a}$. Let $\varphi_t(u)$ denote for the flow of X passing through $u\in M$ at $t=0$, i.e. $\begin{cases}\frac{\partial}{\partial t}\varphi_t(u)=X(\varphi_t(u)) & \forall\, t\in(-\varepsilon,\varepsilon)\\ \varphi_0(u)=u\end{cases}$ If necessary, we shrink V such that $\varphi_t(u)$ exists uniquely in V.

No.

Date

By a similar argument as we did in solution to Exercise 5 b) of Chapter 3, we have $\varphi_t(p) = p$, $\varphi_t(q) = q$ for all t where $\varphi_t(\cdot)$ is defined correspondingly. Note that X is a Killing field, thus $\varphi_t : M \longrightarrow M$ is an isometry for each fixed $t \in (-\varepsilon, \varepsilon)$. It follows that $\varphi_t \circ \gamma$ is another geodesic in V (for sufficiently small $t \in (-\varepsilon, \varepsilon)$) joining $\varphi_t(p) = p$ to $\varphi_t(q) = q$. By the uniqueness of geodesics in a normal neighborhood in M connecting two given points, we conclude that $\varphi_t \circ \gamma = \gamma$, i.e. $\varphi_t(r) = r$ for all $t \in (-\varepsilon, \varepsilon)$ whenever $r \in \gamma \subseteq M$. This proves $\gamma \cap V \subseteq N$.

$\boxed{b)}$ If p is an isolated singular point of X, we can choose a sufficiently small neighborhood $V \subseteq M$ of p such that $V \cap N = \{p\}$, which is a zero dimensional submanifold of M.

If p is not an isolated singular point of X, let $V \subset M$ be a sufficiently small normal neighborhood of p such that there exists $q_1 \in V \cap N$, $q_1 \neq p$. Let γ_1 be the radial geodesic joining p to q_1. By a), $\gamma_1 \cap V \subseteq N$ (if we appropriately shrink V, while still keeping V open and containing p). If $V \cap N = \gamma_1$, then we know $V \cap N$ is a 1-dimensional submanifold of M, since the inclusion $i : \gamma_1 \longrightarrow M$ is bijective and is obviously an immersion onto $\gamma_1 \subset M$, and we can choose V sufficiently small and then apply Proposition 3.7 of Chapter 0. Otherwise, if $V \cap N \supsetneqq \gamma_1$, we can choose $q_2 \in V \cap N \setminus \{\gamma_1\}$ and let γ_2 be the radial geodesic joining p to q_2. Let $Q = \{\mu \exp_p^{-1}(q_1) + \lambda \exp_p^{-1}(q_2) \subset T_pM : \mu, \lambda \in \mathbb{R}\}$, and let $N_2 = \exp_p(Q \cap \exp_p^{-1}(V))$. Note that $\exp_p^{-1}(q_1)$, $\exp_p^{-1}(q_2)$ are linearly independent, and since $\varphi_t : M \longrightarrow M$ fixes both γ_1 and γ_2, we have

$$(d\varphi_t)_p(\exp_p^{-1}(q_1)) = \frac{d}{ds}\Big|_{s=0} \varphi_t(\exp_p(s \cdot \exp_p^{-1}(q_1))) = \frac{d}{ds}\Big|_{s=0} \exp_p(s \cdot \exp_p^{-1}(q_1)) = \exp_p^{-1}(q_1).$$

$$(d\varphi_t)_p(\exp_p^{-1}(q_2)) = \frac{d}{ds}\Big|_{s=0} \varphi_t(\exp_p(s \cdot \exp_p^{-1}(q_2))) = \frac{d}{ds}\Big|_{s=0} \exp_p(s \cdot \exp_p^{-1}(q_2)) = \exp_p^{-1}(q_2).$$

Thus for any $v \in Q$, we have (for any $t \in (-\varepsilon, \varepsilon)$ fixed)

$$\begin{aligned}(d\varphi_t)_p v &= (d\varphi_t)_p(\mu \cdot \exp_p^{-1}(q_1) + \lambda \exp_p^{-1}(q_2)) = \mu (d\varphi_t)_p(\exp_p^{-1}(q_1)) + \lambda (d\varphi_t)_p(\exp_p^{-1}(q_2)) \\ &= \mu \exp_p^{-1}(q_1) + \lambda \exp_p^{-1}(q_2) = v, \quad \text{i.e. } (d\varphi_t)_p = Id : T_pM \longrightarrow T_pM.\end{aligned}$$

It follows that $\varphi_t(\exp_p(s \cdot v)) = \exp_p(s \cdot v)$, $\forall v \in T_pM$, in other words $\varphi_t(q) = q$ for all $q \in N_2$. Differentiating with respect to t, we obtain $X(q) = 0$ for all $q \in N_2$, or equivalently $N_2 \subset N$. By our construction of N_2 one sees easily that $N_2 \subseteq V$, and hence $N_2 \subset V$. Hence $N_2 \subset V \cap N$. If $V \cap N = N_2$, then we are done since N_2 is a 2-dimensional submanifold of M (one can easily see that the inclusion $i: N_2 \to M$ is an immersion at p, and we can shrink V once again so that we can apply Proposition 3.7 of Chapter 0). Otherwise, if $V \cap N \supsetneqq N_2$, we can choose $q_3 \in (V\cap N) \setminus N_2$ and repeat the steps above. Since $\dim T_pM < \infty$, this procedure terminates in at most $\dim T_pM = \dim M$ steps. We can now conclude that $V \cap N$ is a submanifold of M.

Remark: In assertion that "$(d\varphi_t)_p = Id : T_pM \to T_pM$ implies $\varphi_t(\exp_p(s\cdot v)) = \exp_p(s\cdot v)$ for all $v \in T_pM$" we used the local uniqueness of solutions to an ODE system.

$\boxed{c)}$ If $\dim N_k = \dim M$, then $\operatorname{codim} N_k = 0$, which is even. In the follows we assume $\dim N_k < \dim M$.

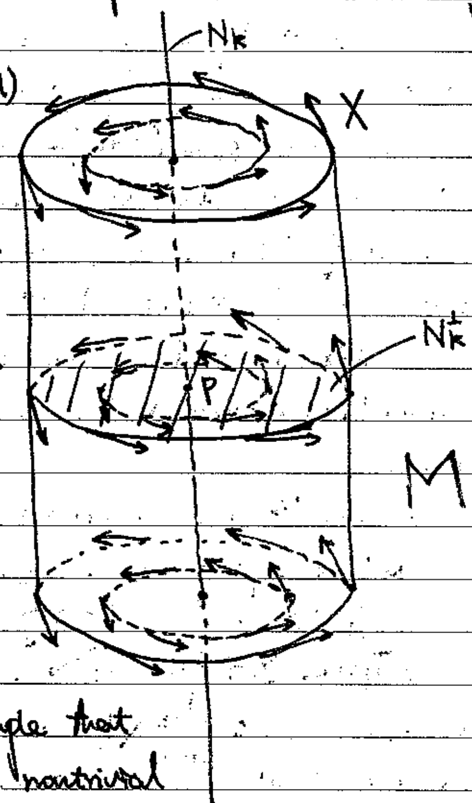
Let $E_p = (T_pN_k)^\perp$ and let $V \subset M$ be a normal neighborhood of p. Since $\dim N_k < \dim M$ by our assumption, E_p is non-trivial. Set $N_k^\perp = \exp_p(E_p \cap \exp_p^{-1}(V))$. Since X is Killing, $\varphi_t : M \to M$ is an isometry for each fixed $t \in (-\varepsilon, \varepsilon)$. Hence for any $Y \in T_pN_k$, $Z \in E_p = (T_pN_k)^\perp$, we have

$0 = \langle Y, Z \rangle_p^M = \langle (d\varphi_t)_p Y, (d\varphi_t)_p Z \rangle_{\varphi_t(p)}^M = \langle Y, (d\varphi_t)_p Z \rangle_p^M$ where we used the observation established in b) that $(d\varphi_t)_p : T_pN_k \to T_pN_k$ is the identity map, along with an earlier observation that $\varphi_t(p) = p$ for all $t \in (-\varepsilon, \varepsilon)$. This proves that $d\varphi_t|_{E_p}$ maps into $E_{\varphi_t(p)} = E_p$. In other words, for all $v \in E_p \cap \exp_p^{-1}(V)$, $\frac{d}{ds}\Big|_{s=0} \varphi_t(\exp_p(s\cdot v)) \in E_p$, or equivalently that $\varphi_t(\exp_p(s\cdot v)) \subset N_k$ for all $t \in (-\varepsilon,\varepsilon)$ and $s \in \mathbb{R}$ such that $s\cdot v \in E_p \cap \exp_p^{-1}(V)$. Since every point in N_k can be parametrized by $\exp_p(s\cdot v)$ for some $s \in \mathbb{R}$, $v \in E_p$ satisfying $s \cdot v \in E_p \cap \exp_p^{-1}(V)$, this proves that $\varphi_t|_{N_k}$ is a diffeomorphism which maps into N_k for all $t \in (-\varepsilon, \varepsilon)$. It follows immediately that X is tangent to $N_k^\perp$, or equivalently that we can view X as an element in $\mathfrak{X}(N_k^\perp)$. Note that, by the definition of $N_k^\perp$, $p \in N_k^\perp$ is an isolated singular point of the Killing vector field $X \in \mathfrak{X}(N_k^\perp)$. By Exercise 5 b) in Chapter 3, X is tangent to the geodesic spheres in $N_k^\perp$ centered at p. Thus we obtain a non-vanishing differentiable vector field on a sphere of dimension $\dim N_k^\perp - 1$. By the theorem cited in the hint (cf. M.A. Armstrong, P198 Theorem 9.6), the dimension of the geodesic spheres centered at p in $N_k^\perp$ must be odd, and thus $\dim N_k^\perp = \left(\begin{array}{c}\text{dimension of the geodesic spheres centered at } p \\ \text{in } N_k^\perp\end{array}\right) + 1$ must be even.

No.

Date

Remark: A good picture to be kept in mind while doing this problem.

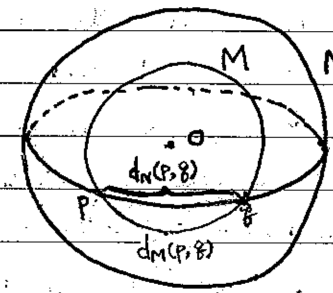


The vector field $X \in \mathfrak{X}(M)$ is generated by the rotations along the axis of the infinite cylinder M. Since the rotation is an isometry for M, X is Killing. N is the rotational axis in the cylinder, and $N^\perp$ are the slices orthogonal to the rotational axis. It is intuitive why X must be tangent to $N_k^\perp$.

This is also a typical example that a diffeomorphism may have nontrivial differential at a fixed point (which seems quite obvious though).

1. If M, N are Riemannian manifolds such that the inclusion $i: M \subset N$ is an isometric immersion, show by an example that the strict inequality $d_M > d_N$ can occur.

• Solution: Let $N = S^2 \subset \mathbb{R}^3$ and M be any parallel on S^2 which is not an equator (or any great circle).



2. Let $\tilde{M}$ be a covering space of a Riemannian manifold M. Show that it is possible to give $\tilde{M}$ a Riemannian structure such that the covering map $\pi: \tilde{M} \to M$ is a local isometry (this metric is called the covering metric). Show that $\tilde{M}$ is complete in the covering metric if and only if M is complete.

• Proof: For each $p \in \tilde{M}$ and $X, Y \in T_p\tilde{M}$, define a Riemannian metric on $\tilde{M}$ by $\langle X, Y \rangle_p^{\tilde{M}} = \langle d\pi_p(X), d\pi_p(Y) \rangle_{\pi(p)}^{M}$. Since $\pi: \tilde{M} \to M$ is a smooth covering map, it is a local diffeomorphism and thus $\langle\ ,\ \rangle_p^{\tilde{M}}$ depends smoothly on $p \in \tilde{M}$ in the sense of Definition 2.1 in Chapter 1. Moreover, since $d\pi_p: T_p\tilde{M} \to T_{\pi(p)}M$ is a vector space isomorphism, $\langle\ ,\ \rangle_p^{\tilde{M}}$ is a well-defined inner product on $T_p\tilde{M}$ for each $p \in \tilde{M}$. It is immediate to see that the covering map $\pi: \tilde{M} \to M$ becomes a local isometry after we equip $\tilde{M}$ with the covering metric constructed above. In the follows we always assume $\tilde{M}$ is equipped with the covering metric.

Next we show that $\tilde{M}$ is complete if and only if M is complete. Let $\gamma: (-\varepsilon, \varepsilon) \to M$ be a geodesic with $\gamma(0) = p$ and $\gamma'(0) = v$. For any $\bar{p} \in \pi^{-1}(p)$, by the unique lifting property of covering spaces, there is a unique lift $\bar{\gamma}: (-\varepsilon, \varepsilon) \to \tilde{M}$, i.e. $\pi \circ \bar{\gamma} = \gamma$ with $\bar{\gamma}(0) = \bar{p}$. Since $\pi: \tilde{M} \to M$ is a local isometry, the inverse locally exists and is also a isometry, and hence $\bar{\gamma}$ is a geodesic in $\tilde{M}$. If M is complete, γ will exist for all time and thus $\bar{\gamma}$ will exist for all time. As all geodesics in $\tilde{M}$ must be of the form $\bar{\gamma}$, this shows that all geodesics in $\tilde{M}$ exist for all time. Conversely, if $\tilde{M}$ is complete, then $\bar{\gamma}$ can be extended to be defined for all time. Then $\pi \circ \bar{\gamma}$ is a geodesic in M which is defined for all time that extends γ. Thus M is geodesically complete.

No.

Date

3. Let $f: M_1 \longrightarrow M_2$ be a local diffeomorphism of a manifold M_1 onto a Riemannian manifold M_2. Introduce on M_1 a Riemannian metric such that f is a local isometry. Show by an example that if M_2 is complete, M_1 need not be complete.

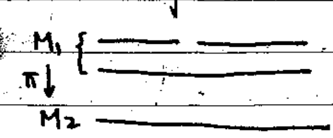
- Proof: Since f is a local diffeomorphism, for any $p \in M_1$, there exists an open neighborhood $\mathcal{U}$ of p in M_1 and an open neighborhood V of $f(p)$ in M_2 such that $f: \mathcal{U} \longrightarrow V$ is a diffeomorphism. As an open submanifold of M_2, V has a natural Riemannian metric $\tilde{h} = h|_V$, thus the pullback $\tilde{g} = f^*\tilde{h}$ is a Riemannian metric on $\mathcal{U}$. By the arbitrariness of choices of $p \in M_1$, this proves that we have a Riemannian metric on M_1. It is immediate to see that f is a local isometry from (M_1, f^*h) to (M_2, h). Indeed, for all $p \in M_1$, df_p is an isomorphism from T_pM_1 to $T_{f(p)}M_2$. For any $u, v \in T_pM_1$, define

$$\langle u, v \rangle_p = \langle df_p(u), df_p(v) \rangle_{f(p)}$$

or equivalently in notation $g_p(u,v) = h_{f(p)}(df_p(u), df_p(v))$. Obviously $g_p(u,v) = g_p(v,u)$; moreover, $\langle u, u \rangle_p = \langle df_p(u), df_p(u) \rangle_{f(p)} \geq 0$ with equality holds if and only if $df_p(u) = 0 \Leftrightarrow u = 0 \in T_pM$ since df_p is an isomorphism. The smoothness of g follows from the smoothness of h and df. Hence g is a Riemannian metric on M_1 with $h_{f(p)}(df_p(u), df_p(v)) = g_p(u,v)$, $\forall\, u, v \in T_pM$, which means that f is a local isometry.

To see an example desired in the problem, let $M_1 = (-2\pi n, 2\pi n)$ for some $n \in \mathbb{N}$, $n > 1$, $M_2 = S^1 \subseteq \mathbb{C}$, $f: M_1 \longrightarrow M_2$ defined by $f(x) = e^{ix}$. Equipping M_1, M_2 with the canonical Riemannian metrics respectively, f is easily verified to be a local isometry, but by the Hopf-Rinow Theorem one easily sees that M_2 is geodesically complete while M_1 is not.

Remark 1: There are tons of examples which satisfy our needs for this example. In fact, any example of a local diffeomorphism which is not a covering map can be adapted to serve as such an example. For example, the covering on the right picture (need to pick M_2 such that it is geodesically complete), say,



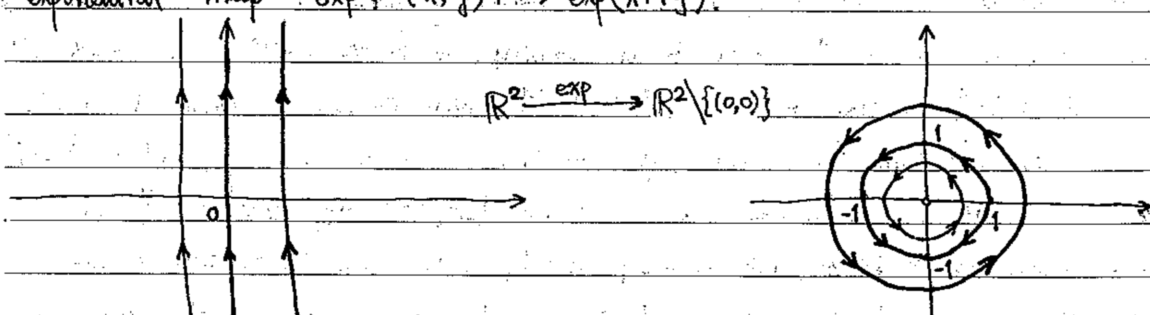
$\pi: S^2 \setminus \{\text{North Pole}\} \longrightarrow \mathbb{RP}^2$, where π is induced from the canonical projection from S^2 to $\mathbb{RP}^2$, with appropriately chosen Riemannian metrics (canonical ones suffice).

Remark 2. It seems that we actually don't need $f: M_1 \to M_2$ to be surjective in the original problem.

4. Consider the universal covering $\pi: M \longrightarrow \mathbb{R}^2 - \{(0,0)\}$ of the Euclidean plane minus the origin. Introduce the covering metric on M (cf. Exercise 2). Show that M is not complete and not extendible, and that the Hopf–Rinow theorem is not true for M (this shows that the definition of non-extendibility, though natural, is not a satisfactory one).

Hint: The single difficult point is to prove that M is non-extendible. Suppose to the contrary, that is, $M \subset M'$, is an isometry and $M \neq M'$. Let $p' \in M'$ be a point in the boundary of M and let $W' \subset M'$ be a convex neighborhood of p'. If we prove that $W' - \{p'\} \subset M$, we obtain a contradiction, since then $\pi(W' - \{p'\}) = \mathcal{U}$ is a neighborhood of $(0,0) \in \mathbb{R}^2$ and considering the closed circle in $\mathcal{U}$ with center at $(0,0)$ we can lift the closed circle into M, which is impossible. To prove that $W' - \{p'\} \subset M$, observe first that through any point $p \in M$ passes a unique geodesic in M that cannot be extended for all values of $t \in \mathbb{R}$. Let $x \in W' \cap M$ and join x to p' by a geodesic $\tilde{\gamma}$ of M'. $\tilde{\gamma}$ coincides initially with a geodesic of M and therefore is the unique geodesic passing through x which cannot be extended for all values of t. From uniqueness and from the fact that p' is a point of the boundary it follows that all the points of $\tilde{\gamma} \cap W'$, except p', belong to M, because $\tilde{\gamma}$ approaches the boundary of M arbitrarily close. Finally, if $z \in W'$ and $z \notin \tilde{\gamma}$, the geodesic joining z to x is, from the uniqueness above, entirely in M, hence $z \in M$.

• $\boxed{0^\circ}$ What is M? $M = \mathbb{R}^2$, and the covering map is given by the standard exponential map $\exp$, $(x, y) \longmapsto \exp(x + iy)$.



This map is basically "wrapping each vertical axis on a circle centered at $(0,0)$".

$\boxed{1^\circ}$ That M is not complete in the covering metric follows easily from Exercise 2, since $\mathbb{R}^2 \setminus \{(0,0)\}$ equipped with the standard Euclidean metric induced from that of $\mathbb{R}^2$ is obviously not geodesically complete.

No.

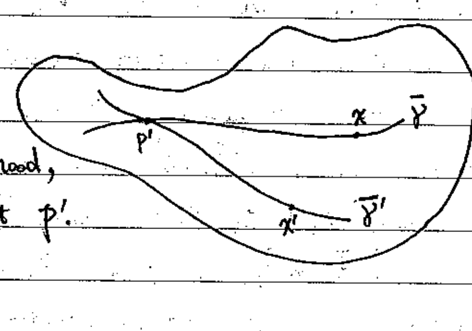
Date

$\boxed{2^\circ}$ We want to show that M is not extendible. Suppose, to the contrary, that M is isometric to a proper open subset of another Riemannian manifold M'. Without loss of generality, we may assume M' is connected, since M is connected. Thus $M' \supset \partial M \neq \phi$ (otherwise $\bar{M} = M \cup \partial M = M$, then M is both closed and open in M', contradicting our assumption that $M \neq \bar{M}$ and that $\bar{M}$ is connected). Let $p' \in \partial M \subset M'$, and let $W' \subset M'$ be a convex (totally normal) neighborhood of p' in $\bar{M}$. First we are going to show that $W' \setminus \{p'\}$ is contained in M, and then we will derive a contradiction out of it.

- To prove that $W' \setminus \{p'\} \subset M$, one observe that through any point $p \in M$ passes a unique geodesic in M that cannot be extended for all values of $t \in \mathbb{R}$. To see this, note that through any point $q \in \mathbb{R}^2 \setminus \{(0,0)\}$ there is a unique geodesic that cannot be extended for all values of $t \in \mathbb{R}$, which is precisely the straight line passing through q and $(0,0)$. Since $\pi : M \longrightarrow \mathbb{R}^2 \setminus \{(0,0)\}$ is a local isometry, for any $p \in M$ one can lift a geodesic in $\mathbb{R}^2 \setminus \{(0,0)\}$ starting from $\pi(p)$ uniquely up into a geodesic in M. Recall as we proved in solution to Exercise 2 of Chapter 7, a geodesic in $\mathbb{R}^2 \setminus \{(0,0)\}$ can be extended for all values of $t \in \mathbb{R}$ if and only if its (unique) lift in M can be extended for all values of $t \in \mathbb{R}$, hence the existence and uniqueness of a geodesic passing through any fixed point $p \in M$ that cannot be extended for all values of $t \in \mathbb{R}$ follows. Now let $x \in W' \cap M$ and join x to p' by a geodesic $\tilde{\gamma}$ of M'. Since M is open in M', there is an open neighborhood $\mathcal{U}$ of x contained in $W' \cap M$, and thus $\tilde{\gamma}$ coincides initially with a geodesic of M, by the uniqueness of geodesics in a convex neighborhood. By our previous discussion, since (W, δ) is a totally normal neighborhood in M' with $0 < \delta < +\infty$, and since the inclusion $i : M \hookrightarrow M'$ is an isometry, $\tilde{\gamma}$ passes through p' in finite time implies that $\tilde{\gamma} \cap M$ is the unique geodesic in M which cannot be extended for all values of $t \in \mathbb{R}$. We claim that $(W' \cap \tilde{\gamma}) \setminus \{p'\} \subset M$, i.e. p' is the only point on $\tilde{\gamma} \cap W'$ which does not lie in M. To see this, first note that, for any $z \in W' \setminus \tilde{\gamma}$, the geodesic connecting x to z lies entirely in M: since the intersection of this geodesic with a neighborhood of x in M is actually a geodesic in M, and since $\tilde{\gamma}$ is the only geodesic in M which cannot be extended for all values of $t \in \mathbb{R}$, the geodesic connecting x to z with $z \notin \tilde{\gamma}$ can be extended for all values of $t \in \mathbb{R}$ while remaining itself within M, by the uniqueness of geodesics in a convex neighborhood. Thus it is immediate that $z \in M$ for any $z \in W' \setminus \tilde{\gamma}$, or equivalently speaking,

$[(M'\setminus M)\cap W'] \subset \bar{\gamma}$. Moreover, since our choice of $x \in W'\cap M$ is arbitrary, we can choose any other $x' \in W'\cap M$, $x' \neq x$, and repeat the argument above to conclude that $[(M'\setminus M)\cap W'] \subset \bar{\gamma}'$, where $\bar{\gamma}'$ is a geodesic in M' joining x' to p'. There are two cases here:

(i) If there exists such an $x' \in (W'\cap M)\setminus\bar{\gamma}$, then by the uniqueness of geodesics joining two distinct points in a totally normal neighborhood, $\bar{\gamma}$ and $\bar{\gamma}'$ intersect only at one single point p'.



According to our previous arguments,

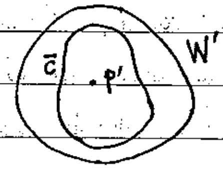
$$\left.\begin{array}{l}[(M'\setminus M)\cap W'] \subset \bar{\gamma} \\ [(M'\setminus M)\cap W'] \subset \bar{\gamma}'\end{array}\right\rangle \Rightarrow [(M'\setminus M)\cap W'] \subset (\bar{\gamma}\cap\bar{\gamma}') = \{p'\} \subset [(M'\setminus M)\cap W'].$$

Thus $(M'\setminus M)\cap W' = \{p'\}$, i.e. $W'\setminus\{p'\} \subset M$.

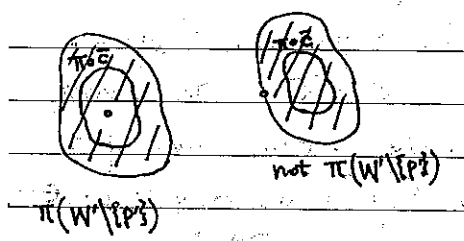
(ii) If there is no $x' \in (W'\cap M)\setminus\bar{\gamma}$, then $(W'\cap M)\setminus\bar{\gamma} = \phi \Rightarrow W'\cap M \subset \bar{\gamma}$, contradicting our assumption that M is open in M' (which implies that $W'\cap M$ is an open subset of M', but $\bar{\gamma}\cap W'$ is only of dimension 1, while any open subset of M' must have dimension equaling $\dim M' = \dim M = 2$.)

Concluding (i) and (ii), we know that $(W'\cap\bar{\gamma})\setminus\{p'\} \subset M$ and $W'\setminus\bar{\gamma} \subset M$, from which we obtain $W'\setminus\{p'\} \subset M$. (Indeed we directly showed $W'\setminus\{p'\} \subset M$ by first auxilliarily proving $W'\setminus\bar{\gamma} \subset M$).

- Now that $W'\setminus\{p'\} \subset M$, we are ready to demonstrate a contradiction.



Step 1. Since M' is a 2-dimensional manifold, we can choose W' sufficiently "small" such that W' is homeomorphic to a pathly connected open subset of $\mathbb{R}^2$ (since W' is pathly connected in M'). For example, by the construction of convex neighborhoods in Proposition of Chapter 3 (pp.76-77), we know that W' can actually chosen to be a geodesic ball $B_\beta(p') = \exp_{p'}(B_\beta(0))$ for some sufficiently small $\beta > 0$, and $\exp_{p'}: T_{p'}M' \supset B_\beta(0) \longrightarrow B_\beta(p')$ is actually a diffeomorphism. It is easily verifield that $\exp_{p'}|_{B_\beta(0)\setminus\{0\}}: B_\beta(0)\setminus\{0\} \longrightarrow B_\beta(p')\setminus\{p'\}$ is also



a diffeomorphism. Thus $\pi_1(W'\backslash\{p'\}) = \pi_1(B_\rho(p')\backslash\{p'\}) = \pi_1(B_\rho(0)\backslash\{0\}) = \mathbb{Z}$, and we can choose a closed loop $\bar{c}$ in $W'\backslash\{p'\}$ such that $[\bar{c}]$ represents a non-trivial element in $\pi_1(W'\backslash\{p'\})$. For simplicity of notations, write $\mathbb{R}^2\backslash\{(0,0)\} \supset \mathcal{U} := \pi(W'\backslash\{p'\})$ and $\mathcal{U} \supset c := \pi\circ\bar{c}$, where $\pi : M \longrightarrow \mathbb{R}^2\backslash\{(0,0)\}$ is the covering map.

Step 2. We make two important observations: $\mathcal{U}$ is bounded in $\mathbb{R}^2\backslash\{(0,0)\}$, and $c = \pi\circ\bar{c}$ is not null-homotopic in $\mathcal{U}$.

(i) $\mathcal{U}$ is bounded in $\mathbb{R}^2\backslash\{(0,0)\}$: For any $q_1, q_2 \in \mathcal{U}$, and any piecewise differentiable curve a joining q_1 to q_2, there exists $\bar{q}_1, \bar{q}_2 \in W'\backslash\{p'\}$ and a piecewise differentiable curve $\bar{a}$ joining $\bar{q}_1$ to $\bar{q}_2$ satisfying $\pi(\bar{q}_1) = q_1$, $\pi(\bar{q}_2) = q_2$, $\pi\circ\bar{a} = a$. (This can be done through the unique lifting property of a covering map: first arbitrarily choose $\bar{q}_1 \in \pi^{-1}(q_1)\cap(W'\backslash\{p'\})$, then uniquely lift a to $\bar{a}$, finally fix $\bar{q}_2 \in \pi^{-1}(q_2)\cap(W'\backslash\{p'\})$.)

Since $\pi : M \longrightarrow \mathbb{R}^2\backslash\{(0,0)\}$ is a local isometry, we always have $\text{length}(a) \le \text{length}(\bar{a})$.

On the other hand, for any $\bar{q}_1, \bar{q}_2$ arbitrarily chosen in $W'\backslash\{p'\}$ and any piecewise differentiable curve joining $\bar{q}_1$ to $\bar{q}_2$ in $W'\backslash\{p'\}$, we can find a curve in $\mathcal{U}$ joining $\pi(\bar{q}_1)$ to $\pi(\bar{q}_2)$ with the unique lift in $W'\backslash\{p'\}$ being precisely the initial curve in $W'\backslash\{p'\}$ joining $\bar{q}_1$ to $\bar{q}_2$. Therefore,

$$d(q_1, q_2) = \inf_{\substack{a \text{ piecewise differentiable} \\ \pi\circ a \text{ joining } q_1 \text{ to } q_2}} \text{length}(\pi\circ a) \le \inf_{\substack{a \text{ piecewise differentiable} \\ \pi\circ a \text{ joining } q_1 \text{ to } q_2}} \text{length}(a) \le \inf_{\substack{a \text{ piecewise differentiable} \\ a \text{ joining } \bar{q}_1 \text{ to } \bar{q}_2}} \text{length}(a) = d(\bar{q}_1, \bar{q}_2) \le \underbrace{\text{some constant}}_{\text{only depending on } W'\backslash\{p'\}}.$$

It then follows from the boundedness of $W'\backslash\{p'\}$ in M that $\mathcal{U}$ is bounded in $\mathbb{R}^2\backslash\{(0,0)\}$.

(ii) $c = \pi\circ\bar{c}$ is not null-homotopic in $\mathcal{U}$: Otherwise, there exists a homotopy $F : [0,1]\times\mathcal{U} \longrightarrow \mathcal{U}$ such that $F(0,t) = c(t)$, $F(1,t) = Id_q$ for some $q \in \mathcal{U}$. By the homotopy lifting property of the covering map (cf. A. Hatcher, "Algebraic Topology", Proposition 1.30 in Section 1.3, pp.60), F lifts to a homotopy in $W'\backslash\{p'\}$ from $\bar{c}$ to a constant loop $Id_{\bar{q}}$, where $\bar{q} \in (W'\backslash\{p'\})\cap\pi^{-1}(q)$, which contradicts our choice of $\bar{c}$, since $\bar{c}$ is null-homotopic in $W'\backslash\{p'\}$.

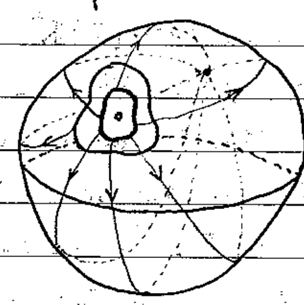
Concluding (i) and (ii), we know that $\mathcal{U}$ can only be of the form $V\backslash\{(0,0)\}$, where $V \subset \mathbb{R}^2$ is an open subset of $\mathbb{R}^2$ containing $(0,0) \in \mathbb{R}^2$ in its interior, since any bounded open subset $\mathcal{O}$ of $\mathbb{R}^2\backslash\{(0,0)\}$ is induced from a bounded open subset $\tilde{\mathcal{O}}$ of $\mathbb{R}^2$, and $\mathcal{O}$ contains a non-trivial loop if and only if $(0,0) \in \tilde{\mathcal{O}}$. In other words, $\pi(W'\backslash\{p'\}) = \mathcal{U}$ is a neighborhood of $(0,0) \in \mathbb{R}^2$. For an illustration for the configuration of $\mathcal{U}$ in $\mathbb{R}^2\backslash\{(0,0)\}$, see the picture on the left bottom on last page.

Step 3. Observe that a loop in $\mathcal{U}$ is null-homotopic in $\mathcal{U}$ if and only if it is null-homotopic in $\mathbb{R}^2\setminus\{(0,0)\}$, since $\mathbb{R}^2\setminus\{(0,0)\}$ deformation retracts to $\mathcal{U}$. This deformation retract does exist, because in Step 2 we showed that $\mathcal{U}$ is a neighborhood of $(0,0)\in\mathbb{R}^2$. It now follows that $c=\pi\circ\bar{c}$ is not null-homotopic in $\mathbb{R}^2\setminus\{(0,0)\}$ because we showed in Step 2 that it is not null-homotopic in $\mathcal{U}$. (This step turns out to be the most crucial in deriving the contradiction.)

Step 4. By the uniqueness in the path lifting property of covering spaces, we know that $c\subset\mathcal{U}$ lifts to $\bar{c}\subset W\setminus\{p'\}\subset M$. Since $\pi: M\longrightarrow\mathbb{R}^2\setminus\{(0,0)\}$ is the universal covering of $\mathbb{R}^2\setminus\{(0,0)\}$, M is simply connected, and thus $\bar{c}$ is null-homotopic in M. Hence $c=\pi\circ\bar{c}$ is also null-homotopic in $\mathbb{R}^2\setminus\{(0,0)\}$, which contradicts our observation in Step 3 that c is not null-homotopic in $\mathbb{R}^2\setminus\{(0,0)\}$. This contradiction completes our proof that M is not extendible.

$\boxed{3^\circ}$ The Hopf-Rinow Theorem is not true for M, since when we were showing "$W\setminus\{p'\}\subset M$" in 2° we have already pointed out that through any point $p\in M$ passes a (unique) geodesic in M that cannot be extended for all values of $t\in\mathbb{R}$. This completes the whole proof for this problem. $\blacksquare$

Remark: The main idea is this: a non-null-homotopic loop in a punctured open neighborhood in a simply-connected space shrinks to a point through a path outside the neighborhood. But this can not be done in $\mathbb{R}^2\setminus\{(0,0)\}$, as pointed out in Step 3.



No.

Date

5. A divergent curve in a Riemannian manifold M is a differentiable mapping $\alpha : [0,\infty) \longrightarrow M$ such that for any compact set $K \subset M$ there exists $t_0 \in (0,\infty)$ with $\alpha(t) \notin K$ for all $t > t_0$ (that is, α "escapes" every compact set in M). Define the length of a divergent curve by $\lim_{t\to\infty} \int_0^t |\alpha'(t)|\, dt$. Prove that M is complete if and only if the length of any divergent curve is unbounded.

- Proof: By the Hopf-Rinow Theorem, we know that

M is geodesically complete $\Longleftrightarrow$ For any $p \in M$, there exists a sequence of compact subsets $K_n \subset M$, $K_n \subset K_{n+1}$ and $\bigcup_{n=1}^{\infty} K_n = M$, such that if $q_n \notin K_n$ then $d(p, q_n) \longrightarrow \infty$.

- If M is geodesically complete, then we can find for $\alpha(0) \in M$ a sequence of compact subsets $K_n \subset M$ as above, where α is any divergent curve in M. By definition, there exists a sequence of strictly increasing real numbers $0 < t_1 < t_2 < \cdots < t_n < t_{n+1} < \cdots < \infty$ such that $\alpha(t_n) \notin K_n$. It follows that

$$\lim_{t\to\infty} \int_0^t |\alpha'(t)|\, dt > \int_0^{t_n} |\alpha'(t)|\, dt \geq d(\alpha(0), \alpha(t_n)) \longrightarrow \infty \quad \text{as } n \longrightarrow \infty.$$

Since the definition of $\lim_{t\to\infty} \int_0^t |\alpha'(t)|\, dt$ is independent of $n \in \mathbb{N}$, the formula above implies that the length of the divergent curve α is unbounded.

- If the length of any divergent curve in M is unbounded, we show that any geodesic $\gamma : [0,\varepsilon) \longrightarrow M$ can be extended for all values of $t \in \mathbb{R}$. Without loss of generality, it suffices to show that γ can be extended for all $[0,\infty)$. Assume the contrary, i.e. $A_\gamma := \{a \in \mathbb{R}^+ : \gamma \text{ can be extended to } [0,a)\}$ has an upper bound $b < +\infty$. Note that $A_\gamma \neq \phi$ since $\varepsilon > 0$ is in A_γ. By the axiom of real numbers, $b^* := \sup A_\gamma$ exists. By definition of b^*, for all sufficiently small $\delta > 0$ (such that $b^* - \delta > 0$), γ can be extended to $[0, b^* - \delta)$. We claim that γ can be actually extended to $[0, b^*)$. To see this, let $\{t_n\}_{n=1}^{\infty}$ be an arbitrary sequence of real numbers in A_γ such that $t_n \longrightarrow b^*$ as $n \to \infty$. Since $A_\gamma \subset \mathbb{R}$, $\{t_n\}_{n=1}^{\infty}$ is a Cauchy sequence in $\mathbb{R}$, and hence $\{\gamma(t_n)\}_{n=1}^{\infty}$ is a Cauchy sequence in M because

$$d(\gamma(t_n), \gamma(t_m)) = \left| \int_{t_n}^{t_m} |\gamma'(t)|\, dt \right| = |\gamma'(0)| \cdot |t_m - t_n| \longrightarrow 0 \quad \text{for all } m, n \to \infty.$$

Let us defined a parametrized version of γ by setting $\tilde{\gamma} : [0, +\infty) \longrightarrow M$ as

$$\tilde{\gamma}(t) := \gamma(b^*(1 - e^{-t})).$$

Then $\tilde{\gamma}$ is defined for all $t \in [0, +\infty)$. Since $\tilde{\gamma}$ is simply a parametrization of γ, it has the same length as that of γ: $\text{length}(\tilde{\gamma}) = \text{length}(\gamma) = \int_0^{b^*} |\gamma'(t)|\, dt = |\gamma'(0)| \cdot b^* < \infty.$

By our assumption that the length of any divergent curve is unbounded, we can conclude that $\tilde{\gamma}$ is not a divergent curve. Hence there exists a compact set $K \subset M$ such that $\tilde{\gamma}(t) \in K$ for all $t \in [0, +\infty)$, i.e. $\tilde{\gamma}$ is contained in K. Since $\tilde{\gamma}$ is simply a reparametrization of γ, this implies that γ is also contained in K. Now we know that $\{\gamma(t_n)\}_{n=1}^{\infty}$ is a Cauchy sequence contained in a compact subset K of M, which implies the existence of a limiting point $l \in M$ for the sequence $\{\gamma(t_n)\}_{n=1}^{\infty}$. Let $W \subset M$ be a totally normal neighborhood of $l \in M$, then obviously $W \cap \gamma \neq \phi$, and we can choose $\gamma(t_N) \in \{\gamma(t_n)\}_{n=1}^{\infty}$ lying in $W \cap \gamma$. Since W is a totally normal neighborhood of l, there exists a geodesic β passing through $\gamma(t_N)$ and l. By the uniqueness of geodesics in W, β and γ coincide in W, which means that γ can be extended through l. Now let us define $\gamma(b^*) = l$, $\gamma'(b^*) =$ the tangent vector of β at b^*, and $\gamma : (b^*-\delta, b^*+\delta) \longrightarrow W$ being the geodesic in W passing through l with prescribed $\gamma'(b^*)$ at b^*. Then the uniqueness of geodesics in W ensures that $\gamma : (b^*-\delta, b^*+\delta) \longrightarrow M$ coincides with the segment of β passing through l and wherever β is defined, and $\gamma : [0, b^*-\delta/2) \rightarrow M$ coincides with the segment of β from $\gamma(t_N)$ to l and wherever $\gamma|_{(b^*-\delta, b^*+\delta)}$ and β are correspondingly defined. Hence $\gamma|_{[0,b^*-\delta/2)} \cup \gamma|_{(b^*-\delta, b^*+\delta)}$ is well-defined as a smooth curve in M, and is actually an extension of γ from $[0, b^*-\delta/2)$ to $[0, b^*+\delta)$. In particular, this gives $b^*+\delta \in A_\gamma$, where $\delta > 0$, and thus $b^* = \sup A_\gamma \geqslant b^*+\delta > b^*$, which is a contradiction. This contradiction implies that $A_\gamma : \{a \in \mathbb{R}^+ : \gamma \text{ can be extended to } [0, a)\}$ is unbounded, or equivalently speaking that any geodesic γ in M can be extended for all values of $t \in \mathbb{R}$.

No.

Date

6. A geodesic $\gamma : [0,\infty) \longrightarrow M$ in a Riemannian manifold M is called a ray starting from $\gamma(0)$ if it minimizes the distance between $\gamma(0)$ and $\gamma(s)$, for any $s \in (0,\infty)$. Assume that M is complete, non-compact, and let $p \in M$. Show that M contains a ray starting from p.

- Proof: Since M is complete but non-compact, there exists a sequence of points $\{q_i\}_{i=1}^{\infty}$ such that $d(p,q_i) \longrightarrow \infty$ as $i \to \infty$, for otherwise M is bounded, and then by the Hopf-Rinow Theorem M is complete and thus closed, which again by the Hopf-Rinow Theorem implies that M is compact, a contradiction. Let $v_i \in T_pM$ be such that $|v_i| = 1$ and $\sigma_i(t) = \exp_p(tv_i)$, $t \in [0, d(p,q_i)]$ be a minimizing geodesic joining p to q_i, whose existence is guaranteed by the completeness of M. By the compactness of the closed unit sphere in T_pM, by possibly passing to a subsequence we may assume $v_i \longrightarrow v \in T_pM$ as $i \to \infty$ for some $v \in T_pM$, $|v| = 1$. Define a geodesic $\gamma(t) = \exp_p(tv)$, $t \in [0,+\infty)$. By the continuity of geodesics with respect to initial conditions (cf. ODE formulation in Chapter 3 (pp. 62-63), and also Remark 3.8 of Chapter 3 on p.72 (related in some other aspects)), σ_i converges to γ pointwise as $i \to +\infty$. Thus by the continuity of the distance function one has that $d(\gamma(0),\gamma(s)) = d(p,\gamma(s)) = \lim_{i\to+\infty} d(p,\sigma_i(s)) = \lim_{i\to+\infty} s = s$, since σ_i's are minimizing geodesics past $s \in \mathbb{R}$ for sufficiently large $i \in \mathbb{N}$. Noting that the length of γ between $\gamma(0)$ and $\gamma(s)$ equals s since $|\gamma'(0)| = |v| = 1$, we know that γ minimizes the distance between $\gamma(0)$ and $\gamma(s)$ for any $s \in (0,\infty)$. This proves that γ is a ray in M starting from $p \in M$.

7. Let M and $\bar{M}$ be Riemannian manifolds and let $f: M \longrightarrow \bar{M}$ be a diffeomorphism. Assume that $\bar{M}$ is complete and that there exists a constant $c>0$ such that $|v| \geq c\,|df_p(v)|$, for all $p \in M$ and all $v \in T_p(M)$. Prove that M is complete.

- Proof: We show that (M, d) is complete as a metric space. Once this is shown, the Hopf-Rinow Theorem asserts that M is geodesically complete.

First, we show that for any $p, q \in M$ one has $d_M(p,q) \geq c \cdot d_{\bar{M}}(f(p), f(q))$. Indeed, for any piecewise differentiable curve γ in M joining p to q, $f \circ \gamma$ is a piecewise differentiable curve in $\bar{M}$ joining $f(p)$ to $f(q)$. Note that

$$\text{length}(\gamma) = \int_0^a |\gamma'(t)|\, dt \geq \int_0^a c \cdot |df_{\gamma(t)}(\gamma'(t))|\, dt = c \cdot \int_0^a |(f\circ\gamma)'(t)|\, dt = c \cdot \text{length}(f\circ\gamma),$$

thus:

$$d_M(p,q) = \inf_{\substack{\gamma(0)=p \\ \gamma(a)=q}} \text{length}(\gamma) \geq c \cdot \inf_{\substack{\gamma(0)=p \\ \gamma(a)=q}} \text{length}(f\circ\gamma) \geq c \cdot \inf_{\substack{\beta(0)=f(p) \\ \beta(a)=f(q)}} \text{length}(\beta) = c \cdot d_{\bar{M}}(f(p), f(q)).$$

Next, we show that (M, d_M) is complete as a metric space. To see this, let $\{x_n\}_{n=1}^{\infty}$ be a Cauchy sequence in M, then $\{f(x_n)\}_{n=1}^{\infty}$ is a Cauchy sequence in $\bar{M}$, since $d_{\bar{M}}(f(x_n), f(x_m)) \leq \frac{1}{c} \cdot d_M(x_n, x_m) \longrightarrow 0$ as $m, n \to \infty$, because $c > 0$.

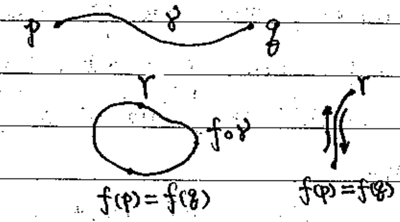
Note that $\bar{M}$ is geodesically complete, by the Hopf-Rinow Theorem $(\bar{M}, d_{\bar{M}})$ is complete as a metric space. Thus $f(x_n) \longrightarrow \bar{q} \in M$ as $n \to \infty$ for some $\bar{q} \in \bar{M}$. Let $q := f^{-1}(\bar{q}) \in M$. Since f is a diffeomorphism, $f(x_n) \to \bar{q} = f(q)$ as $n \to \infty$ implies that $x_n \to q$ as $n \to \infty$. This proves that every Cauchy sequence in M has a limit in M, i.e. (M, d_M) is complete as a metric space.

No.

Date

8. Let M be a complete Riemannian manifold, $\bar{M}$ a connected Riemannian manifold, and $f: M \longrightarrow \bar{M}$ a differentiable mapping that is locally an isometry. Assume that any two points of $\bar{M}$ can be joined by a unique geodesic of $\bar{M}$. Prove that f is injective and surjective (and, therefore, f is a global isometry).

• Proof: $\boxed{1°}$ Injectivity. Assume there exist $p, q \in M$ such that $f(p) = f(q) \in \bar{M}$, $p \neq q$. Since M is complete, there exists a geodesic γ in M joining p to q. Since $f: M \to \bar{M}$ is locally an isometry, $f\circ\gamma$ is a geodesic in $\bar{M}$. But $f(p) = f(q)$ implies that $f\circ\gamma$ is a closed curve in $\bar{M}$, unless $f\circ\gamma$ is a single point in $\bar{M}$. Since f is a local isometry, $f\circ\gamma$ cannot be a single point in $\bar{M}$, thus $f\circ\gamma$ is a closed geodesic in $\bar{M}$ and there exists some $r \in f\circ\gamma$, $r \neq f(p) = f(q)$, and there are two geodesics in $\bar{M}$ joining $f(p)$ to r. By our hypothesis, there is a unique geodesic of $\bar{M}$ joining $f(p)$ to r, hence the image of $f\circ\gamma$ is a closed path in $\bar{M}$ joining $f(p) = f(q)$ to r. Again this contradicts our assumption that f is a local isometry if we exame the behavior of f in a neighborhood of some $\tilde{r} \in f^{-1}(r) \cap \gamma$ in M.



$\boxed{2°}$ Surjectivity. Pick $p \in M$ and let $\bar{p} := f(p) \in \bar{M}$. Define $A := \{\bar{q} \in \bar{M} : \bar{q} = f(q)$ for some $q \in M\}$. Then $A \neq \phi$ since $\bar{p} \in \bar{M}$. Since $f: M \to \bar{M}$ is locally an isometry, A is obviously open. We want to show that A is also closed in $\bar{M}$. To see this, let $\{\bar{q}_i = f(q_i)\}_{i=1}^{\infty}$ be a Cauchy sequence in $\bar{M}$. By assumption, there exist geodesics $\bar{\sigma}_i$ such that $\bar{\sigma}_i$ joins $\bar{p}$ to $\bar{q}_i$, for each $i \in \mathbb{N}$. Let σ_i be a geodesic in M starting from $p \in M$ with initial tangent vector $\sigma_i'(0) = (df_p)^{-1}(\bar{\sigma}_i'(0))$, for all $i \in \mathbb{N}$, where the existence of $\sigma_i'(0)$ is guaranteed by the local isometric property of f at $p \in M$. Moreover, we have $|\sigma_i'(0)| = |(df_p)^{-1}(\bar{\sigma}_i'(0))|$, thus without loss of generality we may assume $|\sigma_i'(0)| = 1 = |\bar{\sigma}_i'(0)|$ for all $i \in \mathbb{N}$. By the local uniqueness of geodesics in $\bar{M}$, one easily checks that $f\circ\sigma_i$ is a geodesic in $\bar{M}$ which coincides with $\bar{\sigma}_i$ wherever they are both defined, since $(f\circ\sigma_i)'(0) = df_p(\sigma_i'(0)) = \bar{\sigma}_i'(0)$ and $(f\circ\sigma_i)(0) = f(p) = \bar{p} = \bar{\sigma}_i(0)$. By possibly passing to subsequences twice, we may assume without loss of generality that $\sigma_i'(0) \longrightarrow v \in T_pM$, $\bar{\sigma}_i'(0) \longrightarrow \bar{v} \in T_{\bar{p}}\bar{M}$ with $|v| = 1 = |\bar{v}|$. By the continuity of $df_p: T_pM \longrightarrow T_{\bar{p}}\bar{M}$, actually we have $\bar{v} = df_p(v)$. Let γ be a geodesic in M starting from p with initial tangent vector $\gamma'(0) = v$, and let $\bar{\gamma}$ be a geodesic in $\bar{M}$ starting from $\bar{p}$

with initial tangent vector $\bar{\gamma}'(0) = \bar{v} = df_p(v)$. Again by the local uniqueness of geodesics in $\bar{M}$, we know that $f \circ \gamma$ coincides with $\bar{\gamma}$ in $\bar{M}$ wherever they are both defined. Now that M is geodesically complete, every geodesics in M is defined for all values in $\mathbb{R}$. Note that, since f is locally an isometry, by our construction we have for all $i \in \mathbb{N}$:
length of the segment of σ_i from p to q_i = length of the segment of $\bar{\sigma}_i$ from $\bar{p}$ to $\bar{q}_i$.
Moreover, if there exists another geodesic η_i in M joining p to q_i, then $f \circ \eta_i$ is a geodesic in $\bar{M}$ joining $\bar{p}$ to $\bar{q}_i$, and by our hypothesis we must have $f \circ \eta_i$ coinciding with $f \circ \sigma_i$ in $\bar{M}$ wherever they are both defined. In particular, this implies that $\eta_i'(0) = \sigma_i'(0)$ in T_pM, and thus η_i coincides with σ_i in M in a sufficiently small neighborhood of $p \in M$. By the smoothness of η_i and σ_i, a bootstrap argument yields $\eta_i = \sigma_i$ in M as geodesic segments joining p to q_i. This proves that σ_i is the only geodesic in M joining p to q_i. By the completeness of M and the Hopf-Rinow Theorem, there exists a minimizing geodesic joining p to q_i. Hence the uniqueness we showed above implies that σ_i is exactly the minimizing geodesic joining p to q_i, from which we obtain

$$d_M(p,q_i) = \text{length}\left(\sigma_i\big|_{[0,d_M(p,q_i)]}\right) = \text{length}\left(\bar{\sigma}_i\big|_{[0,d_{\bar{M}}(\bar{p},\bar{q}_i)]}\right) = d_{\bar{M}}(\bar{p},\bar{q}_i)$$

where the last equality follows from the uniqueness of geodesics joining $\bar{p}$ to $\bar{q}_i$ in $\bar{M}$. Recall that $\{\bar{q}_i\}_{i=1}^{\infty}$ is a Cauchy sequence in $\bar{M}$, thus by the continuity of $d_{\bar{M}}$ we know that $\{d_{\bar{M}}(\bar{p},\bar{q}_i) = d_M(p,q_i)\}_{i=1}^{\infty}$ is indeed a Cauchy sequence in $\mathbb{R}$. by the completeness of $\mathbb{R}$, one has $\lim_{i\to\infty} d_M(p,q_i) = d = \lim_{i\to\infty} d_{\bar{M}}(\bar{p},\bar{q}_i)$ for some $d \in \mathbb{R}$. Now we are eligible for defining $q := \gamma(d) \in M$ and $\bar{q} = f(q) \in \bar{M}$. Let $W \subset \bar{M}$ be a sufficiently small totally normal neighborhood of $\bar{q}$ in $\bar{M}$, such that $W = f(V)$ for some neighborhood V of q in M, and $f|_V : V \to W$ is an isometry. It follows that $d_M(q_i,q) = \text{length of}\left(\gamma\big|_{[d_M(p,q_i),d]}\right) = |d_M(p,q_i) - d| \longrightarrow 0$ as $i \to \infty$, and hence $q_i \in V$ for all sufficiently large $i \in \mathbb{N}$. Note that $f|_V : V \to W$ is an isometry, $f(q_i) = \bar{q}_i \in W$ for all sufficiently large $i \in \mathbb{N}$, and since W is totally normal, we have

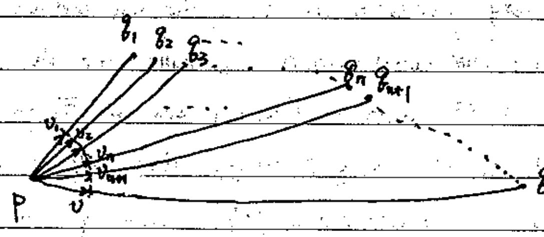
$$d_{\bar{M}}(\bar{q}_i,\bar{q}) = \text{length}(\text{unique minimizing geodesic joining } \bar{q}_i \text{ to } \bar{q}) = \text{length}\left((f\circ\gamma)\big|_{[d_{\bar{M}}(\bar{p},\bar{q}_i),d]}\right)$$
$$= \text{length}\left(\gamma\big|_{[d_M(p,q_i),d]}\right) = d_M(q_i,q) \longrightarrow 0 \text{ as } i \to \infty,$$

where we used the observation that $f \circ \gamma$ is a geodesic in $\bar{M}$ and particularly that $(f\circ\gamma)\big|_{[d_M(p,q_i),d]} = (f\circ\gamma)\big|_{[d_{\bar{M}}(\bar{p},\bar{q}_i),d]}$ is the unique geodesic segment in $W \subset \bar{M}$ joining $\bar{q}_i$ to $\bar{q}$. Therefore $\bar{q} \in \bar{M}$ is the limiting point of the sequence $\{\bar{q}_i\}_{i=1}^{\infty}$ in $\bar{M}$, and $\bar{q} \in A$ since $\bar{q} = f(q)$ for $q \in M$ by definition. Hence A is both open and closed in $\bar{M}$, and

No.

Date

it follows from the connectedness of $\bar{M}$ that $A = \bar{M}$, i.e. $f(M) = \bar{M}$. In other words, we have shown that f is surjective.



9. Consider the upper half-plane $\mathbb{R}^2_+ = \{(x,y) \in \mathbb{R}^2;\ y>0\}$ with the Riemannian metric given by $g_{11} = 1$, $g_{12} = 0$, $g_{22} = \frac{1}{y}$. Show that the length of the vertical segment $x=0$, $\varepsilon \le y \le 1$, with $\varepsilon > 0$, tends to 2 as $\varepsilon \to 0$. Conclude from this that such a metric is not complete. (Observe, nevertheless, that when $y \to 0$ the length of vectors, in this metric, becomes arbitrarily large.)

- Proof: First let us compute Γ^k_{ij}. Recall that $\Gamma^k_{ij} = \frac{1}{2} g^{kl}\left(\frac{\partial g_{li}}{\partial x^j} + \frac{\partial g_{lj}}{\partial x^i} - \frac{\partial g_{ij}}{\partial x^l}\right)$, and note that $g_{11} = 1$, $g_{12} = g_{21} = 0$, $g_{22} = 1/y$. Hence only when $i=j=l=2$ we have $\frac{\partial g_{li}}{\partial x^j} \neq 0$. Easy computation gives $\Gamma^2_{22} = -1/2y$; $\Gamma^k_{ij} = 0$ otherwise. Hence a geodesic $\gamma(t) = (x(t), y(t))$ in $\mathbb{R}^2_+$ under the given metric must satisfy the equations

$$\begin{cases} x''(t) = 0 \\ y''(t) - \dfrac{1}{2y(t)}\left(y'(t)\right)^2 = 0 \end{cases}$$

We can verify that $\begin{cases} x(t) = At + B & A, B \in \mathbb{R} \\ y(t) = C(1-t)^2 & C \neq 0,\ C \in \mathbb{R} \end{cases}$ is a solution to the ODEs above.

Hence if we parametrize the vertical segment $x=0$, $\varepsilon \le y \le 1$, $\varepsilon > 0$ by $x(t) = 0$, $y(t) = (1-t)^2$, $0 \le t \le 1-\sqrt{\varepsilon}$, then $(0, (1-t)^2)$ satisfies the geodesic equations. Moreover, since $|\gamma'(t)| = \sqrt{\langle \gamma'(t), \gamma'(t)\rangle} = \sqrt{\frac{1}{(1-t)^2} \cdot 4(1-t)^2} = 2$ is constant, the parametrization above does give a geodesic in $\mathbb{R}^2_+$. This geodesic $\gamma(t) = (0, (1-t)^2)$, however, obviously can not be extended for all values of $t \in \mathbb{R}$, since $\gamma(1) = (0,0) \notin \mathbb{R}^2_+$. Thus $\mathbb{R}^2_+$ with such a metric is not geodesically complete. (An alternative way to see this is to note that length of the vertical segment $= \int_0^{1-\sqrt{\varepsilon}} |\gamma'(t)|\,dt = 2(1-\sqrt{\varepsilon}) \to 2$ as $\varepsilon \to 0$, thus γ has finite length in $\mathbb{R}^2_+$ with the given metric, and thus cannot be extended for all values of $t \in [0,\infty)$, for otherwise γ could have infinite length. If we use this approach, there is no need to pick a parametrization of γ with constant speed, since length of a curve is independent of parametrizations.)

10. Prove that the upper half-plane $\mathbb{R}^2_+$ with the Lobatchevski metric: $g_{11} = g_{22} = \frac{1}{y^2}$, $g_{12} = 0$, is complete.

- Proof: We show directly that all geodesics in $\mathbb{R}^2_+$ under the Lobatchevski metric extends for all values of $t \in \mathbb{R}$. It suffices to show that a geodesic emanating from a point in $\mathbb{R}^2_+$ can be extended to $[0, +\infty)$.

Recall in Example 3.10 of Chapter 3 (PP.73) we have already known that all geodesics in $\mathbb{R}^2_+$ under the Lobatchevski metric are either rays orthogonal to the x-axis or upper semi-circles orthogonal to the x-axis. Hence it suffices to discuss these two cases.

(i) γ is a straight ray orthogonal to the x-axis. In this case, consider a curve $c: (0, y_0] \longrightarrow \mathbb{R}^2_+$ or $c: [y_0, +\infty) \longrightarrow \mathbb{R}^2_+$ defined by $c(t) = (x_0, t)$. Its parametrization by arc length $\gamma(s)$ is the desired geodesic. Hence it suffices if we prove that $l(c(0, y_0]) = l(c[y_0, +\infty)) = +\infty$ for all $y_0 > 0$. To see this, observe that $c'(t) = (0, 1)$ and hence $|c'(t)|^2 = 1/t^2$. It follows that $l(c(0, y_0]) = \int_0^{y_0} |c'(t)|\, dt = \int_0^{y_0} \frac{dt}{t} = \ln t \Big|_{t=0}^{t=y_0} = +\infty$. Similarly, $l(c[y_0, +\infty)) = \int_{y_0}^{+\infty} |c'(t)|\, dt = \int_{y_0}^{+\infty} \frac{dt}{t} = \ln t \Big|_{t=y_0}^{t=+\infty} = +\infty$. This completes the proof for this case.

(ii) γ is an upper semi-circle orthogonal to the x-axis. In this case, assume the upper semi-circle under consideration is determined by the equation $(x - x_0)^2 + y^2 = r^2$, $y > 0$. Consider the curve $c: (0, \pi) \longrightarrow \mathbb{R}^2_+$ defined by $c(\theta) = (x_0 + r\cos\theta, r\sin\theta)$. Similar to the case (i), the parametrization of c by arc length $\gamma(s)$ is the desired geodesic. Hence it suffices if we show that $l(c(0, \theta_0]) = l(c[\theta_0, \pi)) = +\infty$ for all $\theta_0 \in (0, \pi)$. To see this, observe that $c'(\theta) = (-r\sin\theta, r\cos\theta)$ and hence $|c'(\theta)|^2 = \frac{r^2}{(r\sin\theta)^2} = \frac{1}{\sin^2\theta}$. It follows that $l(c(0, \theta_0]) = \int_0^{\theta_0} |c'(\theta)|\, d\theta = \int_0^{\theta_0} \frac{d\theta}{\sin\theta} = \left[\ln\left(\tan\frac{\theta}{2}\right)\right]_{\theta=0}^{\theta=\theta_0} = +\infty$. Similarly, $l(c[\theta_0, \pi)) = \int_{\theta_0}^{\pi} |c'(\theta)|\, d\theta = \int_{\theta_0}^{\pi} \frac{d\theta}{\sin\theta} = \left[\ln\left(\tan\frac{\theta}{2}\right)\right]_{\theta=\theta_0}^{\theta=\pi} = +\infty$. This completes the proof for this case.

Concluding (i) and (ii), we know that any geodesic in $\mathbb{R}^2_+$ with the Lobatchevski metric extends for all values of $\mathbb{R}$. Hence ($\mathbb{R}^2_+$, Lobatchevski) is geodesically complete.

Remark: One can also determine all geodesics of ($\mathbb{R}^2_+$, Lobatchevski) by directly solving the geodesic equations. Recall that the Christoffel symbols of the Riemannian connection of ($\mathbb{R}^2_+$, Lobatchevski) have been computed in Exercise 8 a) of Chapter 2 (PP.58).

No.

Date

11. Let M be a complete Riemannian manifold, and let X be a differentiable vector field on M. Suppose that there exists a constant $c>0$ such that $|X(p)| \le c$, for all $p \in M$. Prove that the trajectories of X, that is, the curves $\varphi(t)$ in M with $\varphi'(t) = X(\varphi(t))$, are defined for all values of t.

- Proof: Let us fix a point $p \in M$, and denote $\varphi(t,p)$ for the integral curve of X passing through p at $t=0$. By the local existence and uniqueness of linear ODE systems, $\varphi(t,p)$ is at least defined on $[0,\varepsilon)$ for some sufficiently small $\varepsilon>0$. Define $A_p = \{t \in [0,+\infty) : \varphi(t,p) \text{ is defined at } t\}$. Then by the nature of solutions to an ODE system, A_p is open. We want to show that A_p is also closed in $[0,+\infty)$. In fact, let $\{t_n\}_{n=1}^{\infty}$ be a sequence of real numbers in A_p with $\lim_{n\to\infty} t_n = l \in \mathbb{R}$, $l>0$. Then $\{t_n\}_{n=1}^{\infty}$ is a Cauchy sequence in $\mathbb{R}$, and thus $\{\varphi(t_n,p)\}_{n=1}^{\infty}$ is a Cauchy sequence in M since

$$d(\varphi(t_n,p), \varphi(t_m,p)) \le \text{length}\left(\varphi(t,p)\big|_{t\in[t_n,t_m]}\right) = \left|\int_{t_n}^{t_m} \left|\frac{\partial}{\partial t}\varphi(t,p)\right| dt\right| = \left|\int_{t_n}^{t_m} |X(\varphi(t,p))|\, dt\right|$$

$$\le c\,|t_m - t_n| \longrightarrow 0 \quad \text{as } m,n \to \infty.$$

Since M is complete, by the Hopf-Rinow Theorem, M is complete as a metric space. Thus $\varphi(t_n,p) \longrightarrow q \in M$ as $n\to\infty$ for some $q \in M$. Let $U \subset M$ be an open totally normal neighborhood of q in M such that $\varphi(t,q)$ exists for $t \in (-\varepsilon,\varepsilon)$. Pick U sufficiently small such that the uniqueness holds for the ODE system $\begin{cases} \frac{\partial}{\partial t}\varphi(t,q) = X(\varphi(t,q)) \\ \varphi(0,q) = q. \end{cases}$ Then for sufficiently large $n \in \mathbb{N}$, $\varphi(t_n,p) \in U$, and by the uniqueness of integral curves of X passing through q, one has $\varphi(t,p)\big|_{(t_n,l)}$ coinciding with $\varphi(t,q)\big|_{(-\varepsilon,\varepsilon)}$ wherever they are both defined in U. Hence we can define

$$\varphi(t,p) = \begin{cases} \varphi(t,p) & t\in[0,l) \\ \varphi(t-l,q) & t \in [l, l+\varepsilon) \end{cases}$$

It is well-defined since $\varphi(t,p) = \varphi(t-l,q)$ for all $t \in (l-\varepsilon, l)$,

and $\varphi(t,p)$ is the integral curve of X passing through $p \in M$ at $t=0$ since $\varphi(0,p)=p$ and

$$\frac{\partial}{\partial t}\varphi(t,p) = \begin{cases} \frac{\partial}{\partial t}\varphi(t,p) = X(\varphi(t,p)) & \text{for } t\in[0,l) \\ \frac{\partial}{\partial t}\varphi(t-l,q) = X(\varphi(t-l,q)) = X(\varphi(t,p)) & \text{for } t\in[l,l+\varepsilon) \end{cases}$$

this follows directly from our definition of $\varphi(t,p)$ on $t\in[l,l+\varepsilon)$

Therefore, $\varphi(t,p)$ is well-defined at $t=l$, which implies that $l \in A_p$, and hence A_p is closed in $[0,+\infty)$. By the connectedness of $[0,+\infty)$, we know that $A_p = [0,+\infty)$, hence $\varphi(t)$ is defined for all values of $t \in [0,+\infty)$, from which we immediately conclude that $\varphi(t)$ is defined for all values of $t \in \mathbb{R}$.

Remark. The spirit of this proof can be applied to many other circumstances, wherever one needs the existence of a solution on some "complete" domain out of some a priori estimates. Similar proofs can be found in some ODE textbooks regarding the global existence of some solution, or in the proof of the global existence of solutions to Navier-Stokes equations with "small" initial datum. The latter is a good example illustrating the power of this style of arguments for some standard parabolic partial differential equations. In short, "a priori estimates" excludes "blow-ups".

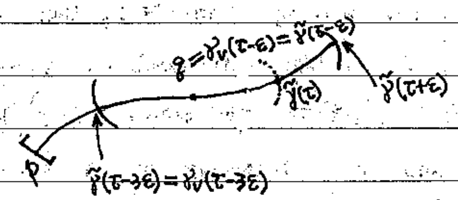
12. A Riemannian manifold is said to be homogeneous if given $p, q \in M$ there exists an isometry of M which takes p into q. Prove that any homogeneous manifold is complete.

- Proof: For any $p \in M$, we show that $\exp_p : T_pM \longrightarrow M$ is defined on all of T_pM.

First, let $\varepsilon > 0$ be such that $B_{2\varepsilon}(p) \subset M$ is a geodesic normal ball centered at p in M. The exponential map is a diffeomorphism from $B_{2\varepsilon}(0) \subset T_pM$ to $\overline{B_{2\varepsilon}(p)} \subset M$. Choose an arbitrary $v \in T_pM$ with $|v| = 1$, and define $\tau := \sup\{t > 0 : \gamma_v(s) = \exp_p(s \cdot v)$ is defined for $s \in [0, t)\}$. Note that $\tau > 0$ since $\tau \geqslant 2\varepsilon > 0$.

We are going to show that $\tau = +\infty$.

If $\tau < +\infty$, by the definition of τ we know that $\gamma_v(s) = \exp_p(sv)$ is defined for $0 \leqslant s \leqslant \tau - \varepsilon$.



By the homogeneous property of M, there exists an isometry $f : M \longrightarrow M$ such that $f(p) = \gamma_v(\tau-\varepsilon) \in M$. Let us define $w := \left(df_{\gamma_v(\tau-\varepsilon)}\right)^{-1}\left(\frac{d}{dt}\Big|_{t=\tau-\varepsilon} \gamma_v(t)\right) \in T_pM$. Obviously $|w| = |v| = 1$.

Let $\gamma_w(t) : (-2\varepsilon, 2\varepsilon) \longrightarrow M$ be a geodesic lying in $B_{2\varepsilon}(p)$ satisfying $\gamma_w(0) = p$, $\gamma_w'(0) = w$.

Define $\tilde{\gamma}(t) = (f \circ \gamma_w)(t - (\tau-\varepsilon)) : (\tau-3\varepsilon, \tau+\varepsilon) \longrightarrow M$, then $\tilde{\gamma}$ is a geodesic in M passing through $f(\gamma_w(\varepsilon))$ at $t = \tau$, because f is an isometry. By the local uniqueness of geodesics in a normal neighborhood, one has $\tilde{\gamma}(t)\big|_{(\tau-3\varepsilon,\tau)} = \gamma_v(t)\big|_{(\tau-3\varepsilon,\varepsilon)}$, and hence the following curve

$$\tilde{\gamma}_v(t) := \begin{cases} \gamma_v(t) & t \in [0, \tau) \\ \tilde{\gamma}(t) & t \in (\tau-3\varepsilon, \tau+\varepsilon) \end{cases}$$

is well-defined. It is easily verified that $\tilde{\gamma}_v(t)$ is an extension of $\gamma_v(t)$ from $[0, \tau-\varepsilon)$ to $[0, \tau+\varepsilon)$. Hence $\tau+\varepsilon \in \{t > 0 : \gamma_v(s)$ is defined for $s \in [0, t)\}$, contradicting our definition of τ. This contradiction tells us that $\gamma_v(s) = \exp_p(sv)$ is defined for all $s \in [0, +\infty)$, and thus any geodesic in M extends for all values of $\mathbb{R}$. Hence M is geodesically complete.

No.

Date

13. Show that the point $p=(0,0,0)$ of the paraboloid $S=\{(x,y,z)\in\mathbb{R}^3;\ z=x^2+y^2\}$ is a pole of S and, nevertheless, the curvature of S is positive.

• Proof: We have already seen in the solution to Exercise 1d) of Chapter 3 that any geodesic passing through the origin $p=(0,0,0)$ must be an origin. Hence $\gamma(t)=(t\cos u_0, t\sin u_0, t^2)$ is a geodesic in S satisfying $\gamma(0)=p$, $\gamma'(0)=(\cos u_0, \sin u_0, 0)\in T_pS$ (of course, $\gamma(t)$ is a geodesic up to reparametrization). Note that $J_1(t)=t\gamma'(t)=(t\cos u_0, t\sin u_0, 2t^2)$ is a Jacobi field along γ satisfying $J_1(0)=0$. Note also that for any $\theta\in\mathbb{R}$ the mapping on S defined by $(v\cos u, v\sin u, v^2)\longmapsto(v\cos(u+\theta), v\sin(u+\theta), v^2)$ is an isometry, thus the vector field generated by this mapping, parametrized as $X:(v\cos u, v\sin u, v^2)\longmapsto(-v\sin u, v\cos u, 0)$, is a Killing vector field on S. By Exercise 8 a) of Chapter 5, the restriction of this Killing field to $\gamma(t)$ is a Jacobi field along γ, i.e. $J_2(t)=X(\gamma(t))=(-t\sin u_0, t\cos u_0, 0)$ is a Jacobi field along γ, which is easily seen to satisfy $J_2(0)=0$. It is also easy to verify that J_1, J_2 are linearly independent, and $J_1(t)\neq 0$, $J_2(t)\neq 0$ for any $t\neq 0$. Recall from Chapter 5 that the linear space of Jacobi fields along γ vanishing at p has dimension equaling $\dim S=2$, thus J_1, J_2 defined above form a basis for this linear space. Since J_1, J_2 are non-vanishing along $\gamma\setminus\{p\}$, there exist no Jacobi fields along γ satisfying $J(p)=0$, $J(q)=0$ for some $q\in\gamma\setminus\{p\}$. This proves that p has no conjugate points along γ, and the arbitrariness of geodesic γ implies that p is a pole of S.

To see that the curvature of S is positive, recall that the sectional curvature of a surface in $\mathbb{R}^3$ coincides with its Gaussian curvature. Parametrize S by $F:\mathbb{R}^2\longrightarrow\mathbb{R}^3$ given by $(u,v)\longmapsto(u,v,F(u,v)=u^2+v^2)$, then $F_u=2u$, $F_v=2v$, $F_{uu}=2$, $F_{uv}=0$, $F_{vv}=2$, and we have (see any differential geometry textbook, or Ng's Homework 6 Problem 3a)):

$$K=\frac{F_{uu}F_{vv}-F_{uv}}{(1+F_u^2+F_v^2)^2}=\frac{4}{(1+4u^2+4v^2)^2}>0 \quad \text{for all } (u,v)\in\mathbb{R}^2.$$

In other words, the curvature of S is positive.

Remark: This is an example showing that poles can exist in non-compact manifolds which have positive sectional curvature (cf. M.P. do Carmo, "Riemannian Geometry", Remark 3.4 in Chapter 7, PP151.). Note that in this example the paraboloid S is a geodesically complete Riemannian manifold. This illustrates the "optimality" of the Hadamard Theorem in some sense.

1. Consider, on a neighborhood in $\mathbb{R}^n$, $n>2$ the metric $g_{ij}=\frac{\delta_{ij}}{F^2}$ where $F\neq 0$ is a function of $(x_1,\cdots,x_n)\in\mathbb{R}^n$. Denote by $F_i=\frac{\partial F}{\partial x_i}$, $F_{ij}=\frac{\partial^2 F}{\partial x_i\partial x_j}$, etc.

a) Show that a necessary and sufficient condition for the metric to have constant curvature K is

$$(*)\quad \begin{cases} F_{ij}=0, & i\neq j,\\ F(F_{jj}+F_{ii})=K+\sum_{i=1}^{n}(F_i)^2. \end{cases}$$

b) Use $(*)$ to prove that the metric g_{ij} has constant curvature K if and only if $F=G_1(x_1)+G_2(x_2)+\cdots+G_n(x_n)$ where $G_i(x_i)=ax_i^2+b_ix_i+C_i$ and $\sum_{i=1}^{n}(4C_ia-b_i^2)=K$.

c) Put $a=K/4$, $b_i=0$, $C_i=1/n$ and obtain the formula of Riemann (see [Ri])

$$(**)\quad g_{ij}=\frac{\delta_{ij}}{\left(1+\frac{K}{4}\sum x_i^2\right)^2}$$

for a metric g_{ij} of constant curvature K. If $K<0$ the metric is defined in a ball of radius $\sqrt{\frac{4}{-K}}$.

d) If $K>0$, the metric $(**)$ is defined on all of $\mathbb{R}^n$. Show that such a metric on $\mathbb{R}^n$ is not complete.

- $\boxed{a)}$ As we have seen in Section 3 of Chapter 8 (P.160-161), one has

$$\Gamma_{ij}^{k}=\delta_{ij}f_k-\delta_{jk}f_i-\delta_{ki}f_j \qquad \text{where } f=\log F,\ f_j=\partial f/\partial x_j, \text{ etc.}$$

Recall that $R_{ijk}^{s}=\partial_j\Gamma_{ik}^{s}-\partial_i\Gamma_{jk}^{s}+\Gamma_{ik}^{m}\Gamma_{jm}^{s}-\Gamma_{jk}^{m}\Gamma_{im}^{s}$ and $R_{ijkl}=g_{ls}R_{ijk}^{s}$. Thus

$$R_{ijkl}=g_{ls}R_{ijk}^{s}=\frac{\delta_{ls}}{F^2}\left(\partial_j\Gamma_{ik}^{s}-\partial_i\Gamma_{jk}^{s}+\Gamma_{ik}^{m}\Gamma_{jm}^{s}-\Gamma_{jk}^{m}\Gamma_{im}^{s}\right)=\frac{1}{F^2}\left(\partial_j\Gamma_{ik}^{l}-\partial_i\Gamma_{jk}^{l}+\Gamma_{ik}^{m}\Gamma_{jm}^{s}-\Gamma_{jk}^{m}\Gamma_{im}^{l}\right),$$

$$\begin{aligned}
&=\frac{1}{F^2}\Big[\partial_j(\delta_{ik}f_l-\delta_{kl}f_i-\delta_{li}f_k)-\partial_i(\delta_{jk}f_l-\delta_{kl}f_j-\delta_{lj}f_k)\\
&\qquad+(\delta_{ik}f_m-\delta_{km}f_i-\delta_{mi}f_k)(\delta_{jm}f_l-\delta_{ml}f_j-\delta_{li}f_m)\\
&\qquad-(\delta_{jk}f_m-\delta_{km}f_j-\delta_{mj}f_k)(\delta_{im}f_l-\delta_{ml}f_i-\delta_{li}f_m)\Big]\\
&=\frac{1}{F^2}(\delta_{ik}f_{jl}-\cancel{\delta_{kl}f_{ji}}-\delta_{li}f_{jk}-\delta_{jk}f_{il}+\cancel{\delta_{kl}f_{ij}}+\delta_{lj}f_{ik})\\
&\quad+\frac{1}{F^2}\Big(\delta_{ik}f_m\delta_{jm}-\delta_{ik}f_m\delta_{ml}f_j-\delta_{ik}\delta_{lj}\sum(f_m)^2-\delta_{km}\delta_{jm}f_if_l+\cancel{\delta_{km}\delta_{ml}f_if_j}+\delta_{km}\delta_{lj}f_if_m\\
&\qquad-\delta_{mi}\delta_{jm}f_kf_l+\delta_{mi}\delta_{ml}f_kf_j+\delta_{mi}\delta_{lj}f_kf_m-\delta_{im}f_m\delta_{jk}f_l+\delta_{jk}f_m\delta_{ml}f_i+\delta_{jk}\delta_{li}\sum(f_m)^2\\
&\qquad+\delta_{km}\delta_{im}f_jf_l-\cancel{\delta_{km}\delta_{ml}f_if_j}-\delta_{km}\delta_{li}f_jf_m+\delta_{mj}\delta_{im}f_kf_l-\delta_{mj}\delta_{ml}f_kf_i-\delta_{mj}f_m\delta_{li}f_k\Big)\\
&=\frac{1}{F^2}\left[\delta_{ik}\left(\frac{F_{jl}}{F}-\frac{F_lF_j}{F^2}\right)-\delta_{li}\left(\frac{F_{jk}}{F}-\frac{F_kF_j}{F^2}\right)-\delta_{jk}\left(\frac{F_{il}}{F}-\frac{F_lF_i}{F^2}\right)+\delta_{lj}\left(\frac{F_{ik}}{F}-\frac{F_kF_i}{F^2}\right)\right]\\
&\quad+\frac{1}{F^2}\Big[\cancel{\delta_{ik}f_jf_l}-\cancel{\delta_{ik}f_lf_j}-\delta_{km}\delta_{jm}f_if_l+\delta_{lj}f_kf_i-\delta_{mi}\cancel{\delta_{jm}f_kf_l}+\delta_{mi}\delta_{ml}f_kf_j+\delta_{lj}f_kf_i\\
&\qquad-\cancel{\delta_{jk}f_if_l}+\cancel{\delta_{jk}f_lf_i}+\delta_{km}\delta_{im}f_jf_l-\delta_{li}f_jf_k+\delta_{mj}\cancel{\delta_{im}f_kf_l}-\delta_{mj}\delta_{ml}f_kf_i-\delta_{li}f_jf_k\Big]\\
&\quad+\frac{1}{F^2}(\delta_{jk}\delta_{li}-\delta_{ik}\delta_{lj})\sum_{m=1}^{n}(f_m)^2\\
&=\frac{1}{F^3}(\delta_{ik}F_{jl}-\delta_{li}F_{jk}-\delta_{jk}F_{il}+\delta_{lj}F_{ik})+\frac{1}{F^4}(\delta_{li}F_kF_j+\delta_{jk}F_lF_i-\delta_{ik}F_lF_j-\delta_{lj}F_kF_i)\\
&\quad+\frac{1}{F^2}(\delta_{jk}\delta_{li}-\delta_{ik}\delta_{lj})\frac{1}{F^2}\sum_{m=1}^{n}(F_m)^2+\frac{1}{F^2}(-\delta_{kj}f_if_l+\cancel{\delta_{lj}f_kf_i}+\cancel{\delta_{il}f_kf_j}+\delta_{lj}f_kf_i+\delta_{ki}f_jf_l-\delta_{li}f_jf_k-\cancel{\delta_{jl}f_kf_i}-\cancel{\delta_{li}f_jf_k})
\end{aligned}$$

No.

Date

$$= \frac{1}{F}\left(g_{ik}F_{jl} - g_{li}F_{jk} - g_{jk}F_{il} + g_{lj}F_{ik}\right) + \cancel{\frac{1}{F^2}\left(g_{li}F_kF_j + g_{jk}F_lF_i - g_{ik}F_lF_j - g_{lj}F_kF_i\right)}$$
$$+ \left(g_{jk}g_{li} - g_{ik}g_{lj}\right)\sum_{m=1}^{n}(F_m)^2 + \cancel{\frac{1}{F^2}\left(-g_{kj}F_iF_l + g_{lj}F_kF_i + g_{ki}F_jF_l - g_{li}F_jF_k\right)}$$
$$= \frac{1}{F}\left(g_{ik}F_{jl} - g_{li}F_{jk} - g_{jk}F_{il} + g_{lj}F_{ik}\right) + \left(g_{jk}g_{li} - g_{ik}g_{lj}\right)\sum_{m=1}^{n}(F_m)^2$$

where we used $\delta_{km}\delta_{jm} = \left\{\begin{matrix} 1 & k=j \\ 2\delta_{kj}=0 & k\neq j \end{matrix}\right\} = \delta_{kj}$, etc.

By Lemma 3.4 of Chapter 4 (P96), we know

$(\mathbb{R}^n, g_{ij} = \delta_{ij}/F^2)$ has constant sectional curvature K

$$\Leftrightarrow R_{ijks} = K(g_{ik}g_{js} - g_{is}g_{jk})$$
$$\Leftrightarrow \frac{1}{F}\left(g_{ik}F_{jl} - g_{li}F_{jk} - g_{jk}F_{il} + g_{lj}F_{ik}\right) + \left(g_{jk}g_{li} - g_{ik}g_{lj}\right)\sum_{m=1}^{n}(F_m)^2 = K\left(g_{ik}g_{js} - g_{is}g_{jk}\right)$$
$$\Leftrightarrow \frac{1}{F}\left(g_{ik}F_{jl} + g_{lj}F_{ik} - g_{li}F_{jk} - g_{jk}F_{il}\right) = \left(K + \sum_{m=1}^{n}(F_m)^2\right)\left(g_{ik}g_{jl} - g_{il}g_{jk}\right) \qquad (\#)$$

Note that $(\#)$ is symmetric in $i \leftrightarrow j$ and $k \leftrightarrow l$.

If $k = i \neq j = l$, $(\#) \Leftrightarrow \frac{1}{F}\left(\frac{F_{jj}}{F^2} + \frac{F_{ii}}{F^2}\right) = \frac{1}{F^4}\left(K + \sum_{m=1}^{n}(F_m)^2\right) \Leftrightarrow F(F_{jj}+F_{ii}) = K + \sum_{m=1}^{n}(F_m)^2$.

If $i=k, j, l$ are three distinct indices, $(\#) \Leftrightarrow \frac{1}{F}\cdot\frac{1}{F^2}F_{jl} = 0 \Leftrightarrow F_{jl} = 0$.

For all other cases, either $(\#)$ trivially holds by "$0=0$", or $(\#)$ is reduced to one of the two cases above by the symmetry previously noted. Therefore, a necessary and sufficient condition for $(\mathbb{R}^n, \delta_{ij}/F^2)$ to have constant sectional curvature K is

$$(*) \quad \left\{\begin{array}{l} F_{ij} = 0, \quad i \neq j, \\ F(F_{jj}+F_{ii}) = K + \sum_{i=1}^{n}(F_i)^2 \end{array}\right.$$

first holds for all $i\neq j$, then obtain validity for all i,j.

$\boxed{b)}$ Since $F_{ij}=0$ for all $i\neq j$, one must have $F_i = F_i(x_i)$. From $F(F_{jj}+F_{ii}) = K + \sum_{i=1}^{n}(F_i)^2$ one obtain $2FF_{jj} = K + \sum_{i=1}^{n}(F_i)^2$, and differentiating both sides gives

$$FF_{jjj} + F_jF_{jj} = F_jF_{jj} \Rightarrow F_{jjj} = 0 \quad \text{for all } j$$

Hence by the Taylor expansion around 0 one obtain $F = G_1(x_1) + \cdots + G_n(x_n)$ where $G_i(x_i) = a_ix_i^2 + b_ix_i + c_i$. Since $F_{ii} = F_{jj}$, $a_i = a_j$ and thus $G_i(x_i) = ax_i^2 + b_ix_i + c_i$.

Finally, $K = 2FF_{jj} - \sum_{i=1}^{n}(F_i)^2 = 4a\sum_{i=1}^{n}(ax_i^2 + b_ix_i + c_i) - \sum_{i=1}^{n}(2ax_i + b_i)^2$

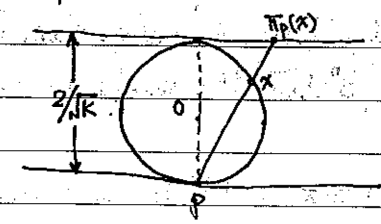
$$= \sum_{i=1}^{n}\left(\cancel{4a^2x_i^2} + \cancel{4ab_ix_i} + 4ac_i - \cancel{4a^2x_i^2} - \cancel{4ab_ix_i} - b_i^2\right) = \sum_{i=1}^{n}(4c_ia - b_i^2).$$

It is easily checked that $F = G_1(x_1) + \cdots + G_n(x_n)$ with $G_i(x_i) = ax_i^2 + b_ix_i + c_i$ and $\sum_{i=1}^{n}(4c_ia - b_i^2) = K$ satisfies $(*)$. This completes the "if and only if" proof.

$\boxed{c)}$ When $a = K/4$, $b_i = 0$, $c_i = 1/n$, we have $F = \sum_{i=1}^{n}\left(\frac{K}{4}x_i^2 + \frac{1}{n}\right) = 1 + \frac{K}{4}\sum_{i=1}^{n}x_i^2$, and hence $g_{ij} = \delta_{ij}\Big/\left(1 + \frac{K}{4}\sum_{i=1}^{n}x_i^2\right)^2$, which establishes $(**)$. If $K<0$, then g_{ij} is defined wherever $F = 1 + \frac{K}{4}\sum_{i=1}^{n}x_i^2 > 0 \Leftrightarrow \sqrt{\sum_{i=1}^{n}x_i^2} < \sqrt{\frac{4}{-K}}$, i.e. the metric g_{ij} is defined in a ball of radius $\sqrt{\frac{4}{-K}}$.

$\boxed{d)}$ If $K>0$, then obviously the metric $(**)$ is defined on all of $\mathbb{R}^n$. Consider the n-sphere in $\mathbb{R}^{n+1}$ with radius $1/\sqrt{K}$, denoted by $S^n(1/\sqrt{K}) := \{(x_1,\cdots,x_{n+1}) \in \mathbb{R}^{n+1} : x_1^2+\cdots+x_n^2+x_{n+1}^2 = \frac{1}{K}\}$.

Let $p = (0,\cdots,0,-1/\sqrt{K}) \in S^n(1/\sqrt{K}) \subset \mathbb{R}^{n+1}$ and $\pi_p : S^n(1/\sqrt{K})\setminus\{p\} \longrightarrow \mathbb{R}^n$ be the stereographic projection from p given by



$$(x_1,\cdots,x_n,x_{n+1}) \xmapsto{\pi_p} \left(\frac{2x_1}{1+\sqrt{K}x_{n+1}}, \cdots, \frac{2x_n}{1+\sqrt{K}x_{n+1}}\right).$$

Since π_p is actually a chart map for $S^n(1/\sqrt{K})$, it is obviously smooth with a smooth inverse (note that π_p is the identity map in coordinates). Let $v_1, v_2 \in T_x(S^n(1/\sqrt{K})\setminus\{p\})$ be two arbitrary tangent vectors at $x=(x_1,\cdots,x_n,x_{n+1}) \in S^n(1/\sqrt{K})\setminus\{p\}$. Note that

$$\sum_{k=1}^n (\pi_p(x))_k^2 = \frac{4\sum_{k=1}^n x_k^2}{(1+\sqrt{K}x_{n+1})^2} = \frac{4(\frac{1}{K}-x_{n+1}^2)}{(1+\sqrt{K}x_{n+1})^2} = \frac{4}{K}\frac{(1+\sqrt{K}x_{n+1})(1-\sqrt{K}x_{n+1})}{(1+\sqrt{K}x_{n+1})^2} = \frac{4}{K}\frac{1-\sqrt{K}x_{n+1}}{1+\sqrt{K}x_{n+1}}$$

$$d\pi_p(v_1) = \begin{bmatrix} \frac{2}{1+\sqrt{K}x_{n+1}} & 0 & \cdots & 0 & -\frac{2\sqrt{K}x_1}{(1+\sqrt{K}x_{n+1})^2} \\ 0 & \ddots & & 0 & \vdots \\ \vdots & & \ddots & \vdots & \\ 0 & \cdots & 0 & \frac{2}{1+\sqrt{K}x_{n+1}} & -\frac{2\sqrt{K}x_n}{(1+\sqrt{K}x_{n+1})^2} \end{bmatrix} \begin{bmatrix} v_1^1 \\ \vdots \\ v_1^{n+1} \end{bmatrix} = \begin{pmatrix} \frac{2v_1^1}{1+\sqrt{K}x_{n+1}} - \frac{2\sqrt{K}x_1 v_1^{n+1}}{(1+\sqrt{K}x_{n+1})^2} \\ \vdots \\ \frac{2v_1^n}{1+\sqrt{K}x_{n+1}} - \frac{2\sqrt{K}x_n v_1^{n+1}}{(1+\sqrt{K}v_{n+1})^2} \end{pmatrix}$$

$$d\pi_p(v_2) = \begin{pmatrix} \frac{2v_2^1}{1+\sqrt{K}x_{n+1}} - \frac{2\sqrt{K}x_1 v_2^{n+1}}{(1+\sqrt{K}x_{n+1})^2} \\ \vdots \\ \frac{2v_2^n}{1+\sqrt{K}x_{n+1}} - \frac{2\sqrt{K}x_n v_2^{n+1}}{(1+\sqrt{K}x_{n+1})^2} \end{pmatrix}$$

thus

$$\langle d\pi_p(v_1), d\pi_p(v_2)\rangle_{(\mathbb{R}^n, g_{ij})}(\pi_p(x)) = \frac{1}{\left(1+\frac{K}{4}\sum_{k=1}^n(\pi_p(x))_k^2\right)^2}\sum_{k=1}^n\left(\frac{2v_1^k}{1+\sqrt{K}x_{n+1}} - \frac{2\sqrt{K}x_k v_1^{n+1}}{(1+\sqrt{K}x_{n+1})^2}\right)\left(\frac{2v_2^k}{1+\sqrt{K}x_{n+1}} - \frac{2\sqrt{K}x_k v_2^{n+1}}{(1+\sqrt{K}x_{n+1})^2}\right)$$

$$= \frac{1}{\left(1+\frac{K}{4}\cdot\frac{4}{K}\cdot\frac{1-\sqrt{K}x_{n+1}}{1+\sqrt{K}x_{n+1}}\right)^2}\left[\frac{4\sum_{k=1}^n v_1^k v_2^k}{(1+\sqrt{K}x_{n+1})^2} - \frac{4\sqrt{K}v_1^{n+1}\sum_{k=1}^n x_k v_1^k + 4\sqrt{K}v_2^{n+1}\sum_{k=1}^n x_k v_2^k}{(1+\sqrt{K}x_{n+1})^3} + \frac{4K v_1^{n+1}v_2^{n+1}\sum_{k=1}^n x_k^2}{(1+\sqrt{K}x_{n+1})^4}\right]$$

$$= \frac{(1+\sqrt{K}x_{n+1})^2}{4}\left[\frac{4\sum_{k=1}^n v_1^k v_2^k}{(1+\sqrt{K}x_{n+1})^2} + \frac{4\sqrt{K}v_1^{n+1}v_2^{n+1}x_{n+1} + 4\sqrt{K}v_2^{n+1}v_1^{n+1}x_{n+1}}{(1+\sqrt{K}x_{n+1})^3} + \frac{4Kv_1^{n+1}v_2^{n+1}\frac{1}{K}(1-Kx_{n+1}^2)}{(1+\sqrt{K}x_{n+1})^4}\right]$$

$$= \sum_{k=1}^n v_1^k v_2^k + \frac{2\sqrt{K}v_1^{n+1}v_2^{n+1}x_{n+1}}{1+\sqrt{K}x_{n+1}} + \frac{v_1^{n+1}v_2^{n+1}(1+\sqrt{K}x_{n+1})(1-\sqrt{K}x_{n+1})}{(1+\sqrt{K}x_{n+1})^2}$$

$$= \sum_{k=1}^n v_1^k v_2^k + v_1^{n+1}v_2^{n+1}\left(\frac{2\sqrt{K}x_{n+1}}{1+\sqrt{K}x_{n+1}} + \frac{1-\sqrt{K}x_{n+1}}{1+\sqrt{K}x_{n+1}}\right) = \sum_{k=1}^n v_1^k v_2^k + v_1^{n+1}v_2^{n+1} = \sum_{k=1}^{n+1} v_1^k v_2^k = \langle v_1, v_2\rangle_{\mathbb{R}^{n+1}}$$

No.

Date

where we used $v_1, v_2 \in T_x(S^n(1/\sqrt{k})\setminus\{p\}) = T_xS^n \Leftrightarrow \langle v_1, x\rangle = \sum_{k=1}^{n+1} v_1^k x_k = 0, \langle v_2, x\rangle = \sum_{k=1}^{n+1} v_2^k x_k = 0$.
Hence $\pi_p : S^n(1/\sqrt{k})\setminus\{p\} \longrightarrow \mathbb{R}^n$ is an isometry, where $S^n(1/\sqrt{k})\setminus\{p\}$ is equipped with the induced (round) metric from $\mathbb{R}^{n+1}$, and $\mathbb{R}^n$ is equipped with the metric g_{ij}.
Since $(S^n(1/\sqrt{k})\setminus\{p\}, \text{round})$ is not geodesically complete, neither is $(\mathbb{R}^n, g_{ij})$.

2. Show that if M^k is a closed, totally geodesic submanifold of H^n, $k \le n$, then M^k is isometric to H^k. Determine all the totally geodesic submanifolds of H^n.

Proof. For an arbitrary point $p \in M^k \subset H^n$, let X, Y be arbitrary linearly independent tangent vectors in T_pM^k. Since M^k is totally geodesic, the second fundamental form vanishes at p for all $\eta \in (T_pM)^\perp$, hence $B(X,Y) = B(X,X) = B(Y,Y) = 0$. By the Gauss equation, one has $\langle \bar{R}(X,Y)X,Y\rangle = \langle R(X,Y)X,Y\rangle - \langle B(Y,Y), B(X,X)\rangle + \langle B(X,Y), B(X,Y)\rangle$
$$= \langle R(X,Y)X,Y\rangle - 0 + 0 = \langle R(X,Y)X,Y\rangle.$$
It follows that $K^{M^k}(X,Y) = K^{H^n}(X,Y) = -1$, for all $p \in M^k$ and all $X, Y \in T_pM^k$ that are linearly independent. Note that H^n is geodesically complete, and that M^k is closed in H^n, it follows from the Hopf-Rinow Theorem that M^k is also geodesically complete. By Theorem 4.1 of Chapter 8 (P163), the universal covering manifold $\tilde{M}$ of M, equipped with the covering metric, is isometric to H^k. To see that M^k is simply connected, fix a point $p \in M^k$ and consider the exponential map $\exp_p : T_pM \to M$. Since M^k is geodesically complete with constantly negative curvature, by Lemma 3.2 in Chapter 7 (P149), $\exp_p : T_pM \longrightarrow M$ is a local diffeomorphism defined on the whole tangent space T_pM. By the Hopf-Rinow Theorem, for arbitrary $q \in M$ such that $p \neq q$, there exists a minimizing geodesic joining p to q which lies entirely in M. Denote γ for this geodesic. Since M is totally geodesic in H^n, γ is also a geodesic in H^n. Note that there is a unique minimizing geodesic $\bar{\gamma}$ joining p to q in H^n (this is because all the geodesics of H^n are of the type described in Proposition 3.1 of Chapter 8: two upper semi-circles centering on $\{(x_1, \cdots, x_n) : x_n = 0\} = \partial H^n$ has at most one intersection point, etc.), and thus $\gamma = \bar{\gamma}$ is the unique geodesic in M joining p to q. This implies that $\exp_p : T_pM \longrightarrow M$ is both surjective and injective. Hence $\exp_p : T_pM \longrightarrow M$ is a bijective local diffeomorphism, thus a diffeomorphism. Since $T_pM^k \cong \mathbb{R}^k$ is simply connected, so is M^k. This proves $M = \tilde{M}$ and hence M^k is isometric to H^k.
Now we determine all the totally geodesic submanifolds of H^n. First note that,

for each integer $2\leqslant k\leqslant n$, the intersections with H^n of k-hyperplanes of $\mathbb{R}^n$, orthogonal to ∂H^n and the intersections with H^n of k-spheres of $\mathbb{R}^n$, with center on ∂H^n, are totally geodesic submanifolds of H^n. To see this, we consider the case for k-planes and k-spheres separately. Each intersection with H^n of a k-hyperplane of $\mathbb{R}^n$ orthogonal to ∂H^n can be naturally identified with H^k. Denote Λ for this intersection. By Proposition 3.1 of Chapter 8 (P162), all geodesics of Λ are either a straight line orthogonal to $\partial\Lambda$ or an upper semicircle orthogonal to $\partial\Lambda$, both contained in a 2-dimensional hyperbolic plane perpendicular to $\partial\Lambda$ sitting in Λ. Since Λ is orthogonal to ∂H^n, any 2-dimensional hyperbolic plane sitting in Λ orthogonal to $\partial\Lambda\subseteq\partial H^n$ is also a 2-dimensional hyperbolic plane sitting in H^n orthogonal to ∂H^n. Again by Proposition 3.1 of Chapter 8 (P162), any geodesic of Λ is also a geodesic of H^n. This implies that the intersection with H^n of a k-hyperplane of $\mathbb{R}^n$ orthogonal to ∂H^n is a totally geodesic submanifold of H^n. For an intersection with H^n of a k-sphere $S^k_R(a)$ of $\mathbb{R}^n$ centering at $a\in\partial H^n$ (denote Σ for this intersection), note that there exists a conformal transformation of $\mathbb{R}^n$ taking H^n onto itself which maps a k-plane of $\mathbb{R}^n$ diffeomorphically to $S^k_R(a)$, and hence the restriction of this conformal transformation to H^n maps the intersection with H^n of the k-plane (which can be chosen to be orthogonal to ∂H^n) diffeomorphically to $S^k_R(a)\cap H^n$. By the Liouville Theorem (since the only non-trivial part of this problem is located at the case $n\geqslant 3$), the restriction of this conformal transformation to H^n is an isometry of H^n, thus $\Sigma=S^k_R(a)\cap H^n$ is the image of some Λ (as denoted above for the intersection of a k-plane of $\mathbb{R}^n$ orthogonal to ∂H^n with H^n) under an isometry on H^n. By Proposition 2.9 of Chapter 6 (P132), the isometry image of a totally geodesic submanifold is also a totally geodesic submanifold, and hence Σ is also totally geodesic in H^n. This proves that the intersection with H^n of a k-sphere of $\mathbb{R}^n$ centering on ∂H^n is a totally geodesic submanifold of H^n. By now we have verified our first observation, in the meanwhile explained the "It is not difficult to verify ..." part of the first paragraph on P177.

Next, we claim that each connected component of a totally geodesic submanifold of H^n is contained in a totally geodesic submanifold of H^n of either "type Λ" or "type Σ" as described above. In fact, let M^k be a k-dimensional connected, totally geodesic submanifold of H^n, and let p be an arbitrary point on M^k.

No.

Date

If T_pM is orthogonal to ∂H^n, there is a k-plane of $\mathbb{R}^n$ orthogonal to ∂H^n which coincides with T_pM; otherwise there is a unique k-sphere of $\mathbb{R}^n$ centering on ∂H^n passing through p and has its tangent plane at p coinciding with T_pM. In other words, there exists a totally geodesic submanifold G of H^n, either of type Λ or of type Σ, such that $p \in M \cap G$ and $T_pM = T_pG$. Note that in either case G is geodesically complete, and let us keep in mind that M is connected. The claim would have been established as long as we show $M \subseteq G$. In fact, let $q \in M$ be a point in a normal neighborhood of p, and let σ be a geodesic segment in M joining p to q. Then $\sigma'(0) \in T_pM = T_pG$, and hence σ is a geodesic of G as long as it remains in G (since G is totally geodesic). Recall that G is geodesically complete, thus σ can be arbitrarily extended in G. By the uniqueness of geodesics in H^n (the Cartan-Hadamard Theorem asserts that the exponential map $\exp_p : T_pH^n \to H^n$ is a diffeomorphism), σ is completely contained in G, and in particular $q \in G$. This proves that p has an open neighborhood of M^k contained in G. Now, for any q in this normal neighborhood of p in M^k, by Exercise 1 of Chapter 2 (P56) we know T_qM is the image of T_pM under the parallel transport of T_pM along a geodesic σ joining p to q. By Exercise 7 of Chapter 6 (P140), the parallel transport along σ in M is the same as the parallel transport of $T_pM \subseteq T_pH^n$ along σ in H^n, and the latter for the same reason is the same as the parallel transport of $T_pG = T_pM$ along σ in G. It follows that $T_qM = T_qG$, and we can repeat the argument above to see that q has an open neighborhood contained in G. In fact we have shown a little more than that: we showed that a geodesic segment starting at p is contained in G as long as it remains in M, and for any point $q \in M$ which can be joined to p via a geodesic segment, a geodesic segment starting at q is contained in G as long as it remains in M. Repeatedly, we know that any point $r \in M$ which can be joined to p via a piecewise geodesic in M with finitely many "broken points" is contained in G. Note that M is connected and the set $\Omega_p := \{x \in M : x$ can be joined to p via a piecewise geodesic in M with finitely many "broken points"$\}$ is both open (by definition) and closed (if $\{\omega_n\}_{n=1}^{\infty} \subset \Omega_p$ and $\omega_n \to \omega_0 \in M$, then ω_0 has a geodesic normal neighborhood $\mathcal{U}$ such that $\omega_m \in \mathcal{U}$ for sufficiently large $m \in \mathbb{N}$), thus $M = \Omega_p \subset G$. This completes the verification of the claim which asserts that each connected component of a totally geodesic submanifold of H^n is contained in a "type Λ" or "type Σ" totally geodesic submanifold of H^n.

Finally, combining all our previous discussions together, we determine all the totally geodesic submanifolds of H^n as follows: a submanifold of H^n is totally geodesic if and only if each of its connected components is contained either in the intersection with H^n of a k-hyperplane of $\mathbb{R}^n$ orthogonal to ∂H^n or the intersection with H^n of a k-sphere of $\mathbb{R}^n$ centering on ∂H^n. Let us point it out that one side of the conclusion follows from our previous discussions while the other direction is trivial.

Remark: In general, a Riemannian manifold need not have any totally geodesic submanifold except for geodesics (which are 1-dimensional totally geodesic submanifolds). The existence of totally geodesic submanifolds implicitly asserts some subtle "symmetry" of the Riemannian manifold: intuitively a totally geodesic submanifold can be viewed as a set of points "formed" by "continuously varying" families of geodesics, but without some "symmetry" this set of points can't form a manifold; for example it may have singularities at intersecting locations of geodesics. In this point of view, space forms have many totally geodesic submanifolds because they have a large group of isometries. This definitely helps: for example, when an isometry has fixed points, the set of fixed points forms a totally geodesic submanifold of the ambient Riemannian manifold.

No.

Date

3. (Another model of the hyperbolic space). Consider on $\mathbb{R}^{n+1}$ the quadratic form

$$Q(x_0, x_1, \cdots, x_n) = -(x_0)^2 + \sum_{i=1}^{n}(x_i)^2, \quad (x_0, \cdots, x_n) \in \mathbb{R}^{n+1}.$$

With the pseudo-Riemannian metric $(\ ,\ )$ induced by Q (Cf. Exercise 9, Chap. 2), $\mathbb{R}^{n+1}$ will be denoted by L^{n+1} (The Lorentzian space). Denote by H^n_k, $k = -\frac{1}{r^2}$, the connected component corresponding to $x_0 > 0$ of the regular surface of $\mathbb{R}^{n+1}$ given by $Q(x) = -r^2$, $r > 0$. (Geometrically $Q(x) = -r^2$ is a hyperboloid of two sheets and H^n_k is the sheet contained in the half-space $x_0 > 0$.)

a) Show that for all $x \in H^n_k$, the vector $\eta = \frac{x}{r}$ is normal to the tangent space $T_x(H^n_k)$.

b) Prove that $(\eta, \eta) = -1$, and that it is possible to choose a basis $b_0, \cdots, b_n$ of L^{n+1} with $b_0 = \eta$, $(b_i, b_j) = \delta_{ij}$, $(b_i, b_0) = 0$, $i, j = 1, \cdots, n$. (Use the fact that the index of a quadratic form does not depend on the basis chosen to represent it.) Conclude that the metric induced by L^{n+1} on H^n_k is Riemannian.

c) Use the pseudo-Riemannian connection $\bar{\nabla}$ of L^{n+1} (Cf. Exercise 9, Chap. 2) to show that $S_\eta = \left(-\frac{1}{r}\right)I$, where I is the identity map. Conclude that $B(X,Y) = \frac{\eta}{r}(X,Y)$, and use the Gauss formula to show that the sectional curvature of H^n_k is constant and equals $k = -\frac{1}{r^2}$.

d) Let $O(1, n+1)$ be the subgroup of linear transformations of $\mathbb{R}^{n+1}$ which preserve the metric $(\ ,\ )$. Show that the elements of $O(1, n+1)$ with $\det > 0$ are isometries of H^n_k and that given $X, Y \in H^n_k$ and orthonormal bases $\{v_i\} \in T_X(H^n_k)$ and $\{w_i\} \in T_Y(H^n_k)$, $i = 1, \cdots, n$, the restriction to H^n_k of the "linear" transformation which takes $\frac{X}{r} \to \frac{Y}{r}$ and $v_i \to w_i$ is an isometry of H^n_k. Conclude then that H^n_k has constant curvature (which we already know from (c)) and that H^n_k is complete.

e) Show that H^n_{-1} is isometric to the hyperbolic space H^n.

f) Show that the symmetries of H^n_k with respect to the plane P which passes through the origin of $\mathbb{R}^{n+1}$ and contains the x_0-axis are isometries of H^n_k. Conclude that all of the geodesics of H^n_k which pass through $(r, 0, \cdots, 0)$ are obtained as intersections $H^n_k \cap P$.

• $\boxed{a)}$ Let $\gamma: (-\varepsilon, \varepsilon) \longrightarrow H^n_k$ be a curve in H^n_k with $\gamma(0) = x$. Write $\gamma(t) = (\gamma_0(t), \cdots, \gamma_n(t))$. Then $-(\gamma_0(t))^2 + (\gamma_1(t))^2 + \cdots + (\gamma_n(t))^2 = -r^2 \Rightarrow -\gamma_0'(0)\gamma_0(0) + \gamma_1'(0)\gamma_1(0) + \cdots + \gamma_n'(0)\gamma_n(0) = 0$, or equivalently that $(x, \gamma'(0)) = ((\gamma_0(0), \cdots, \gamma_n(0)), \gamma'(0)) = 0$. Since $r > 0$, this is further equivalent to $(\eta, \gamma'(0)) = (\frac{x}{r}, \gamma'(0)) = 0$ for all such curve γ in H_k^n with $\gamma(0) = x$. By the arbitrariness of γ, this proves that $\eta = \frac{x}{r}$ is normal to the tangent space $T_x(H_k^n)$.

$\boxed{b)}$ Since $x \in H_k^n$, $(x,x) = -r^2$ and hence $(\eta,\eta) = (\frac{x}{r}, \frac{x}{r}) = \frac{1}{r^2}(x,x) = -\frac{1}{r}\cdot r^2 = -1$. To choose the desired basis, we proceed by mathematical induction. First let $b_0 := \eta$. Denote $\mathcal{U}^1 := \{u \in L^{n+1} : (u, b_0) = 0\}$. If $\mathcal{U}^1 = \{0\}$, then $L^{n+1} = \operatorname{span}\{b_0\}$ and we are done. In the case $\mathcal{U}^1 \neq \{0\}$, let $0 \neq u_1 \in \mathcal{U}^1$ and define $b_1 := \frac{u_1}{(u_1,u_1)}$, which then satisfies $(b_1, b_1) = 1 = \delta_{11}$ and $(b_1, b_0) = 0$. It is easy to check that b_0, b_1 are linearly independent. Now let $\mathcal{U}^2 := \{u \in L^{n+1} : (u, b_0) = 0$ and $(u, b_1) = 0\}$. If $\mathcal{U}^2 = \{0\}$, then $L^{n+1} = \operatorname{span}\{b_0, b_1\}$ and we are done. In the case $\mathcal{U}^2 \neq \{0\}$, let $0 \neq u_2 \in \mathcal{U}^2$ and define $b_2 := \frac{b_2}{(b_2,b_2)}$, which then satisfies $(b_2, b_2) = 1 = \delta_{22}$, $(b_2, b_1) = 0 = \delta_{21}$ and $(b_2, b_0) = 0$. Inductively, one can obtain $b_0, \cdots, b_n$ of L^{n+1} satisfying $b_0 = \eta$, $(b_i, b_j) = \delta_{ij}$, and $(b_i, b_0) = 0$ for $i,j = 1, \cdots, n$. Since $\dim L^{n+1} = n+1 < \infty$, this process must terminate after we obtain $(n+1)$ linearly independent vectors $b_0, \cdots, b_n$ in L^{n+1}. This establishes the existence of a basis as desired. Finally, for any $v_1, v_2 \in T_x(H_k^n)$, we know from a) that $(\eta, v_1) = 0 = (\eta, v_2)$, and hence $v_1, v_2 \in \operatorname{span}\{b_1, \cdots, b_n\}$. Since $(b_i, b_j) = \delta_{ij}$ for $i,j = 1, \cdots, n$ and the index of a quadratic form does not depend on the basis chosen to represent it, $(\,,\,)_{H_k^n}$ is positively definite. Obviously $(\,,\,)_{H_k^n}$ varies smoothly on H_k^n since $(\,,\,)$ varies smoothly on L^{n+1}. Hence we conclude that the metric induced by L^{n+1} on H_k^n is Riemannian.

$\boxed{c)}$ Note that $N(x) = \frac{x}{r}$ is naturally an extension of $\eta(p) \in (T_p H_k^n)^\perp$. By Proposition 2.3 of Chapter 6 on P128, one has for each $X \in T_p H_k^n$ (where $p \in H_k^n$):

$$S_\eta(X) = -(\bar\nabla_X N)^T = -[\bar\nabla_X N - (\bar\nabla_X N)^\perp] = (\bar\nabla_X N, \eta(p)) - \bar\nabla_X N = (\bar\nabla_X N, N) - \bar\nabla_X N$$

$$= X(N,N) - \bar\nabla_X N = -\bar\nabla_X N \qquad \text{where we used } (N,N) \equiv 1 \text{ on } H_k^n.$$

Note that as we have computed in the solution of Exercise 9 in Chapter 2 b) (P58-59), all Christoffel symbols of L^{n+1} vanishes since $g_{ij} = \text{const.}$ for $0 \le i,j \le n$. It follows that

$$S_\eta(X) = -\bar\nabla_X N = -X(N) = -\sum_{i=0}^n X^i \frac{\partial}{\partial x^i}\Big(\sum_{k=1}^{n} \frac{x^k}{r}\frac{\partial}{\partial x^k}\Big) = -\frac{1}{r}\sum_{i,k=0}^n X^i \frac{\partial x^k}{\partial x^i}\frac{\partial}{\partial x^k}$$

$$= -\frac{1}{r}\sum_{i,k=0}^n X^i \delta_{ki}\frac{\partial}{\partial x^k} = -\frac{1}{r}\sum_{k=0}^n X^k \frac{\partial}{\partial x^k} = -\frac{1}{r}X$$

which proves that $S_\eta = -\frac{1}{r}I$, where I is the identity map. It then follows from the definition that, for all $X, Y \in T_p H_k^n$, one has

$(B(X,Y), \eta) = (S_\eta(X), Y) = (-\frac{1}{r}X, Y) = -\frac{1}{r}(X,Y)$. Since $B(X,Y) \in (T_p H_k^n)^\perp = \operatorname{span}\{\eta\}$, we know that $B(X,Y) = (B(X,Y), \eta)\eta = -\frac{\eta}{r}(X,Y)$. Finally, by the Gauss formula,

$$K(X,Y) - \bar{K}(X,Y) = (B(X,X), B(Y,Y)) - (B(X,Y), B(X,Y))$$
$$= \left(\frac{\eta}{r}(X,X), \frac{\eta}{r}(Y,Y)\right) - \left(\frac{\eta}{r}(X,Y), \frac{\eta}{r}(X,Y)\right)$$
$$= \frac{(\eta,\eta)}{r^2}\left[((X,X),(Y,Y)) - ((X,Y),(X,Y))\right]$$
$$= -\frac{1}{r^2}|X\wedge Y|^2 = -\frac{1}{r^2} \text{ whenever } X,Y\in T_pH^n_k \text{ are orthonormal.}$$

Since all the Christoffel symbols of L^{n+1} vanish, $\bar{K}(X,Y)=0$ for all $X,Y\in T_pH^n_k$. It follows that $K(X,Y) = -\frac{1}{r^2}$ for all $X,Y\in T_pH^n_k$ orthonormal. Hence the sectional curvature of H^n_k is constant and equals $k = -\frac{1}{r^2}$.

$\boxed{d)}$ Let $A\in O(1,n+1)$ and $\det A>0$. For each $x\in L^{n+1}$ such that $Q(x) = (x,x) = -r^2$, one has $(Ax,Ax) = (x,x) = -r^2$. Choosing the canonical basis on $\mathbb{R}^{n+1}$, $(\,,\,)$ can be represented by a symmetric matrix $\begin{pmatrix}-1&&&\\&1&&\\&&\ddots&\\&&&1\end{pmatrix}$. Writing $A = \left(\begin{array}{c|c} a_{11} & \mathfrak{a} \\ \hline \beta & \Lambda\end{array}\right)$ where $a_{11}\in\mathbb{R}$, $\mathfrak{a}\in\mathbb{R}^{1\times n}$, $\beta\in\mathbb{R}^{n\times 1}$, $\Lambda\in\mathbb{R}^{n\times n}$. The condition $A\in O(1,n+1)$ is then interpreted as $A^T\begin{pmatrix}-1&&&\\&1&&\\&&\ddots&\\&&&1\end{pmatrix}A = \begin{pmatrix}-1&&&\\&1&&\\&&\ddots&\\&&&1\end{pmatrix}$, or $\left(\begin{array}{c|c} -a_{11}^2+\beta^T\beta & -a_{11}\mathfrak{a}+\beta^T\Lambda \\ \hline -a_{11}\mathfrak{a}^T+\Lambda^T\beta & -\mathfrak{a}^T\mathfrak{a}+\Lambda^T\Lambda\end{array}\right) = \left(\begin{array}{c|c} -1 & 0 \\ \hline 0 & \begin{matrix}1&&\\&\ddots&\\&&1\end{matrix}\end{array}\right)$, i.e.

$$\begin{cases} -a_{11}^2+\beta^T\beta = -1 \\ -a_{11}\mathfrak{a}+\beta^T\Lambda = 0 \\ -\mathfrak{a}^T\mathfrak{a}+\Lambda^T\Lambda = I_n \end{cases}$$

If in addition we assume $a_{11}\det(\Lambda - \frac{1}{a_{11}}\beta\mathfrak{a}) = \det A = 1$, it's not necessarily true that A maps H^n_k to H^n_k, unlike what is claimed by do Carmo. We'll give a counterexample in a remark that follows. The correct assumption is $a_{11}>0$.

We show that if $a_{11}>0$ then A maps H^n_k to H^n_k, not to $(-H^n_k)$. In other words, we assume that
$$\begin{cases} -a_{11}^2+\beta^T\beta = -1 \\ -a_{11}\mathfrak{a}+\beta^T\Lambda = 0 \\ -\mathfrak{a}^T\mathfrak{a}+\Lambda^T\Lambda = I_n \\ a_{11}>0 \end{cases}$$
and are going to show that if $x = \binom{x_0}{y}\in\mathbb{R}^{n+1}$, $x_0\in\mathbb{R}$, $y\in\mathbb{R}^n$, satisfies $\begin{cases}-x_0^2+y^Ty = -r^2 \\ x_0>0\end{cases}$, then $(Ax)_0 = a_{11}x_0 + \mathfrak{a}y > 0$.

To see this, note that $a_{11}x_0+\mathfrak{a}y = a_{11}\sqrt{y^Ty+r^2} + \mathfrak{a}y = \sqrt{a_{11}^2y^Ty + a_{11}^2r^2} + \mathfrak{a}y$ under our assumptions $a_{11}>0$, $x_0>0$. To show that $a_{11}x_0+\mathfrak{a}y>0$, it suffices if we show $a_{11}^2y^Ty + a_{11}^2r^2 - (\mathfrak{a}y)^2>0$. In fact, one has

$a_{11}^2y^Ty + a_{11}^2r^2 - (\mathfrak{a}y)^2 = a_{11}^2y^Ty + a_{11}r^2 - y^T\mathfrak{a}^T\mathfrak{a}y = a_{11}^2y^T(-\mathfrak{a}^T\mathfrak{a}+\Lambda^T\Lambda)y + a_{11}r^2 - y^T\mathfrak{a}^T\mathfrak{a}y =$
$= -a_{11}^2y^T\mathfrak{a}^T\mathfrak{a}y + a_{11}^2y^T\Lambda^T\Lambda y + a_{11}^2r^2 - y^T\mathfrak{a}^T\mathfrak{a}y = -(a_{11}^2+1)y^T\mathfrak{a}^T\mathfrak{a}y + a_{11}^2y^T\Lambda^T\Lambda y + a_{11}^2r^2 =$
$= -y^T\mathfrak{a}^T(-\beta^T\beta-2)\mathfrak{a}y + a_{11}^2y^T\Lambda^T\Lambda y + a_{11}^2r^2 = y^T\mathfrak{a}^T\beta^T\beta\mathfrak{a}y + 2y^T\mathfrak{a}^T\mathfrak{a}y + a_{11}^2y^T\Lambda^T\Lambda y + a_{11}^2r^2 =$
$= (\beta\mathfrak{a}y)^T\beta\mathfrak{a}y + 2(\mathfrak{a}y)^T\mathfrak{a}y + a_{11}^2(\Lambda y)^T\Lambda y + a_{11}^2r^2 > 0.$

Hence $A|_{H^n_k}$ maps H^n_k into H^n_k, and thus is an isometry of H^n_k since the metric

on H^n_k is induced from that on L^{n+1}, which $A \in O(1,n+1)$ preserves.
Note that $\{\frac{X}{r}, v_1, \cdots, v_n\}$, $\{\frac{Y}{r}, \omega_1, \cdots, \omega_n\}$ are two "orthonormal" bases of L^{n+1}, thus any linear transformation A on L^{n+1} taking $E = [\frac{X}{r}; v_1; \cdots; v_n]$ to $F = [\frac{Y}{r}; \omega_1; \cdots; \omega_n]$ satisfies

$$J = \begin{pmatrix} -1 & & \\ & \ddots & \\ & & 1 \end{pmatrix} \quad \begin{cases} E^T J E = J \\ F^T J F = J \\ F = AE \end{cases} \Rightarrow E^T A^T J A E = J = E^T J E \Rightarrow A^T J A = J. \text{ Hence } A \in O(1,n+1).$$

Moreover, since $AX = A(r\cdot\frac{X}{r}) = rA(\frac{X}{r}) = r\cdot\frac{Y}{r} = Y \in H^n_k$, and $A(H^n_k)$ is a connected component of $H^n_k \cup (-H^n_k)$, we have $A(H^n_k) = H^n_k$. Thus $A|_{H^n_k}$ maps H^n_k into H^n_k, and since $A \in O(1,n+1)$, we know that $A|_{H^n_k}$ is an isometry of H^n_k. It follows that for any $X, Y \in H^n_k$ there exists an isometry of H^n_k which takes X to Y. Note that by taking $Y = X$ and $\{\omega_i\}$ an arbitrary permutation of $\{v_i\}$, we have $K(X,\sigma_1) = K(X,\sigma_2)$ for any two dimensional subspaces σ_1, σ_2 of $T_X(H^n_k)$. If $n \geq 3$, this implies H^n_k is isotropic, and by Schur's Theorem (Exercise 8 in Chapter 4, P106) together with the connectivity of H^n_k we know H^n_k has constant sectional curvature. If $n = 2$, the transitivity of the action of the isometry group of H^n_k which we just proved already implies that H^2_k has constant sectional curvature. That H^n_k is geodesically complete follows from Exercise 12 of Chapter 7 (P154).

Remark: We give an example to show that $A \in O(1,n+1)$ and $\det A > 0$ does not imply A is an isometry of H^n_k. Consider the case $n=2$, let $A = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$. Then

$$\begin{cases} \begin{pmatrix} a & c \\ b & d \end{pmatrix}\begin{pmatrix} -1 & \\ & 1 \end{pmatrix}\begin{pmatrix} a & b \\ c & d \end{pmatrix} = \begin{pmatrix} -1 & \\ & 1 \end{pmatrix} \\ \det A = ad - bc = 1 \end{cases} \Rightarrow A = \begin{pmatrix} \sqrt{b^2+1} & b \\ b & \sqrt{b^2+1} \end{pmatrix} \text{ or } A = \begin{pmatrix} -\sqrt{b^2+1} & b \\ b & -\sqrt{b^2+1} \end{pmatrix}.$$

In the former case, for $X = \begin{pmatrix} x_0 \\ y \end{pmatrix} \in \mathbb{R}^2$ with $x_0 > 0$ we have $(AX)_0 = x_0\sqrt{b^2+1} + by = \sqrt{(y^2+r^2)(b^2+1)} + by = \sqrt{(by)^2 + r^2b^2 + y^2 + r^2} + by > 0$ always true. However, in the latter case one has $(AX)_0 = -x_0\sqrt{b^2+1} + by = -\sqrt{(by)^2 + r^2b^2 + y^2 + r^2} + by < 0$ always true. It is geometrically intuitive to see that A maps H^2_k to $(-H^2_k)$ and maps $(-H^2_k)$ to H^2_k. This counterexample shows that the original problem written by do Carmo is incorrect. Indeed, $O(1,n+1)$ has four connected components, of which only the identity component $SO^+(1,n+1)$ is the isometry group of H^n_k. Perhaps do Carmo was thinking about $SO(n)$ in $O(n)$. $\begin{pmatrix} 1 & \\ & 1 \end{pmatrix}$ and $\begin{pmatrix} -1 & \\ & -1 \end{pmatrix}$ are both isometries with positive determinant, but the latter maps H^n_k to H^n_k. In addition to having positive determinant, one needs both time-like orientation preserving and space-like orientation preserving constraints to obtain an isometry of H^n_k.

No.

Date

$\boxed{e)}$ By c) d) we know H^n_{-1} is geodesically complete with constant sectional curvature -1. By Theorem 4.1 in Chapter 8 (P.163), the universal covering of H^n_{-1} with covering metric is isometric to the hyperbolic space H^n. Moreover, since H^n_{-1} is simply connected, it is isometric to the hyperbolic space H^n itself.

$\boxed{f)}$ By the rotational symmetry of H^n_k, it suffices to prove the assertion for a plane P which is spanned by $\{x, e_1\}$ where $x \in H^n_k$ and $\{e_1, \cdots, e_n\}$ is an orthonormal basis of $T_x H^n_k$. If $v, w \in \mathbb{R}^{n+1}$, say $v = b^0 x + b^1 e_1 + \cdots + b^n e_n$ and $w = c^0 x + c^1 e_1 + \cdots + c^n e_n$, then $(\Phi(v), \Phi(w)) = -b^0 c^0 + b^1 c^1 + \cdots + b^n c^n = (v, w)$, where

$$\Phi: \mathbb{R}^{n+1} \longrightarrow \mathbb{R}^{n+1}$$

$$a^0 x + a^1 e_1 + a^2 e_2 + \cdots + a^n e_n \longmapsto a^0 x + a^1 e_1 - a^2 e_2 - \cdots - a^n e_n$$

is the "reflection in the x, e_1 plane". Thus Φ preserves $\{y \in \mathbb{R}^{n+1} \mid (y, y) = -r^2\}$, which has two connected components, $H^n_k = \{y^0 > 0\}$ and $-H^n_k = \{y^0 < 0\}$. By continuity, since $\Phi(x) = x$, Φ must map H^n_k to H^n_k. It's then a diffeomorphism on H^n_k since $\Phi^2 = \mathrm{Id}$. Also, since Φ is linear, $d\Phi_p = \Phi$, and hence for any $v, w \in T_p H^n_k$ where $p \in H^n_k$, one has $(d\Phi_p(v), d\Phi_p(w)) = (\Phi(v), \Phi(w)) = (v, w)$, so Φ is a local isometry, and thus an isometry since it is a diffeomorphism on H^n_k.

Now let γ be a geodesic of H^n_k which passes through $(r, 0, \cdots, 0)$, we claim that γ is contained in a plane spanned by the x_0-axis and $\gamma'(0)$, assuming $\gamma(0) = (r, 0, \cdots, 0)$. In fact, let $\Phi: \mathbb{R}^{n+1} \to \mathbb{R}^{n+1}$ be the reflection with respect to the plane P spanned by the x_0-axis and $\gamma'(0)$. By our previous arguments, Φ is an isometry on H^n_k which obviously fixes $H^n_k \cap P$. Thus γ and $\Phi(\gamma)$ are both geodesics on H^n_k passing through the point $(r, 0, \cdots, 0)$. By the local uniqueness of geodesics, $\gamma = \Phi(\gamma)$ on a small neighborhood on H^n_k around $(r, 0, \cdots, 0)$. Since H^n_k has negatively constant sectional curvature and is simply connected and geodesically complete, by the Cartan-Hadamard Theorem, the exponential map $\exp_p: T_p H^n_k \to H^n_k$ is a diffeomorphism for all $p \in H^n_k$, in particular for $p = (r, 0, \cdots, 0)$. This proves $\gamma = \Phi(\gamma)$ globally, and implies $\gamma \subset H^n_k \cap P$.

actually, one may not need the Cartan-Hadamard Theorem to see $\gamma = \Phi(\gamma)$.

By the geodesic completeness of H^n_k, for any point $q \in H^n_k \cap P$ there exists a geodesic joining $(r, 0, \cdots, 0)$ to q, and by our previous argument, this geodesic must be the segment on $H^n_k \cap P$ which joins $(r, 0, \cdots, 0)$ to q. Hence any point on $H^n_k \cap P$ is contained in γ, which gives $\gamma = H^n_k \cap P$. Hence all of the geodesics of H^n_k which passes through $(r, 0, \cdots, 0)$ are obtained as intersections $H^n_k \cap P$ where P is an appropriate 2-dimensional plane in $\mathbb{R}^{n+1}$.

or equivalently in $\mathbb{L}^{n+1}$

4. Identify $\mathbb{R}^4$ with $\mathbb{C}^2$ by letting (x_1, x_2, x_3, x_4) correspond to $(x_1+ix_2,\ x_3+ix_4)$.
Let $S^3 = \{(z_1, z_2) \in \mathbb{C}^2;\ |z_1|^2+|z_2|^2 = 1\}$, and let $h: S^3 \longrightarrow S^3$ be given by
$h(z_1, z_2) = (e^{\frac{2\pi i}{q}} z_1,\ e^{\frac{2\pi i r}{q}} z_2)$, $(z_1, z_2) \in S^3$, where q and r are relatively prime integers, $q > 2$.

a) Show that $G = \{id, h, \cdots, h^{q-1}\}$ is a group of isometries of the sphere S^3, with the usual metric, which operates in a totally discontinuous manner. The manifold S^3/G is called a lens space.

b) Consider S^3/G with the metric induced by the projection $p: S^3 \longrightarrow S^3/G$. Show that all the geodesics of S^3/G are closed but can have different lengths.

• $\boxed{a)}$ By definition of h, it is immediate to see that $G = \{id, h, \cdots, h^{q-1}\}$ is a group.
To see that G is a group of isometries of the sphere S^3, it suffices if we show that h is an isometry on S^3. For simplicity of notations, let $\alpha := 2\pi/q$ and $\beta = 2\pi r/q$.
Then $h(z_1, z_2) = (e^{i\alpha} z_1,\ e^{i\beta} z_2)$ and $dh_{(z_1,z_2)} = (e^{i\alpha} dz_1,\ e^{i\beta} dz_2)$. For any $p \in S^3$, if $u, v \in T_pS^3$ are given by $\begin{cases} u = (u_1, u_2) = (u_{11}+iu_{12},\ u_{21}+iu_{22}) \\ v = (v_1, v_2) = (v_{11}+iv_{12},\ v_{21}+iv_{22}) \end{cases}$, then one has

$$\begin{aligned}
\langle dh(u), dh(v)\rangle_{h(p)} &= \langle (e^{i\alpha} u_1,\ e^{i\beta} u_2),\ (e^{i\alpha} v_1,\ e^{i\beta} v_2)\rangle_{h(p)} \\
&= \langle e^{i\alpha} u_1,\ e^{i\alpha} v_1\rangle_{h(p)} + \langle e^{i\beta} u_2,\ e^{i\beta} v_2\rangle_{h(p)} \\
&= e^{i\alpha} e^{-i\alpha} \langle u_1, v_1\rangle + e^{i\beta} e^{-i\beta} \langle u_2, v_2\rangle \\
&= \langle u_1, v_1\rangle + \langle u_2, v_2\rangle = \langle u, v\rangle_p,
\end{aligned}$$

which proves that $h: S^3 \longrightarrow S^3$ is an isometry on S^3. (One may also prove this by showing that h is the restriction to S^3 of an isometry on $\mathbb{C}^2 \cong \mathbb{R}^4$.)
Now we show that G acts on S^3 in a totally discontinuous manner. We need to show for any $(z_1, z_2) \in S^3$ that there exists a neighborhood $\mathcal{U}$ of (z_1, z_2) on S^3 such that $h^k(\mathcal{U}) \cap \mathcal{U} = \phi$ except for $h^0 = Id$. In fact, for $k = 1, \cdots, q-1$, if $h^k(z_1, z_2) = (z_1, z_2)$ then one has $e^{ik\cdot\frac{2\pi}{q}} = 1$ and $e^{ik\cdot\frac{2\pi}{q} r} = 1$, or equivalently that $q \mid k$ (since $(q, r) = 1$), which is impossible since $0 < k < q$. By the continuity of h^k (which we definitely needs to show that h is an isometry on S^3, and is obvious to see), there exists an open neighborhood $\mathcal{U}_k$ of (z_1, z_2) in S^3 such that $h^k(\mathcal{U}_k) \cap \mathcal{U}_k = \phi$ (by the Hausdorff property of S^3). Letting $\mathcal{U} := \bigcap_{k=1}^{q-1} \mathcal{U}_k$, then $\mathcal{U} \neq \phi$ since $(z_1, z_2) \in \mathcal{U}$, and $\mathcal{U}$ is an open neighborhood of (z_1, z_2). By our construction, it is obvious to see that $h^k(\mathcal{U}) \cap \mathcal{U} = \phi$ for all $k = 1, \cdots, q-1$, i.e. G acts totally discontinuously on S^3. Thus S^3/G is a well-defined manifold.

No.

Date

(b) All geodesics of S^3/G lift to geodesics of S^3 under the metrics we constructed. Since all geodesics of S^3 are closed, so are all geodesics of S^3/G. To see that geodesics on S^3/G can have different lengths, let γ_1, γ_2 be geodesics on S^3 given by $\gamma_1(t) = (e^{it}, 0)$, $\gamma_2(t) = (0, e^{it})$, $t \in [0, 2\pi)$. Under the projection $p: S^3 \longrightarrow S^3/G$, one has $\text{length}(p(\gamma_1)) = \frac{2\pi}{q}$, $\text{length}(p(\gamma_2)) = \frac{2\pi r}{q}$, which can be different when we have $r \neq 1$.

5. (Connections of conformal metrics). Let M be a differentiable manifold. Two Riemannian metrics g and $\bar{g}$ on M are conformal if there exists a positive function $\mu: M \longrightarrow \mathbb{R}$ such that $\bar{g}(X,Y) = \mu g(X,Y)$, for all $X, Y \in \mathfrak{X}(M)$. Let ∇ and $\bar{\nabla}$ be the Riemannian connections of g and $\bar{g}$, respectively. Prove that $\bar{\nabla}_X Y = \nabla_X Y + S(X,Y)$, where

$$S(X,Y) = \frac{1}{2\mu}\left[(X\mu)Y + (Y\mu)X - g(X,Y)\,\mathrm{grad}\,\mu\right]$$

and $\mathrm{grad}\,\mu$ is calculated in the metric g, that is, $X(\mu) = g(X, \mathrm{grad}\,\mu)$.

Hint: Since $\bar{\nabla}$ is obviously symmetric, it suffices to show that $\bar{\nabla}$ is compatible with $\bar{g}$, that is, that $X(\bar{g}(Y,Z)) = \bar{g}(\bar{\nabla}_X Y, Z) + \bar{g}(Y, \bar{\nabla}_X Z)$. But the first member of the equality above is $X(\mu g(Y,Z)) = X(\mu) g(Y,Z) + \mu g(\nabla_X Y, Z) + \mu g(Y, \nabla_X Z)$, and the second is $\mu g(\nabla_X Y, Z) + \mu g(Y, \nabla_X Z) + \mu[g(S(X,Y), Z) + g(Y, S(X,Z))]$. Therefore, it is enough to prove that $X(\mu) g(Y,Z) = \mu[g(S(X,Y),Z) + g(Y, S(X,Z))]$, which follows from a direct calculation.

Proof: Since $S(X,Y) = S(Y,X)$ by its definition, one has

$\bar{\nabla}_X Y - \bar{\nabla}_Y X = \nabla_X Y - \nabla_Y X + S(X,Y) - S(Y,X) = \nabla_X Y - \nabla_Y X = [X,Y]$ for all $X, Y \in \mathfrak{X}(M)$.

Hence $\bar{\nabla}$ is symmetric. To see that $\bar{\nabla}$ is also metric-compatible, note that

$$\mu[g(S(X,Y),Z) + g(Y, S(X,Z))] = \mu\left[g\left(\tfrac{1}{2\mu}[(X\mu)Y + (Y\mu)X - g(X,Y)\,\mathrm{grad}\,\mu], Z\right) + g\left(Y, \tfrac{1}{2\mu}[(X\mu)Z + (Z\mu)X - g(X,Z)\,\mathrm{grad}\,\mu]\right)\right]$$

$$= \tfrac{1}{2}(X\mu) g(Y,Z) + \tfrac{1}{2}(Y\mu) g(X,Z) - \tfrac{1}{2} g(X,Y) g(\mathrm{grad}\,\mu, Z) + \tfrac{1}{2}(X\mu) g(Y,Z) + \tfrac{1}{2}(Z\mu) g(Y,X) - \tfrac{1}{2} g(X,Z) g(Y, \mathrm{grad}\,\mu)$$

$$= (X\mu) g(Y,Z) + \cancel{\tfrac{1}{2}(Y\mu) g(X,Z)} + \cancel{\tfrac{1}{2}(Z\mu) g(Y,X)} - \cancel{\tfrac{1}{2} g(X,Y)(Z\mu)} - \cancel{\tfrac{1}{2} g(X,Z)(Y\mu)} = (X\mu) g(Y,Z),$$

and hence $X(\bar{g}(Y,Z)) = X(\mu) g(Y,Z) + \mu g(\nabla_X Y, Z) + \mu g(Y, \nabla_X Z)$

$$= \mu g(S(X,Y),Z) + \mu g(Y, S(X,Z)) + \mu g(\nabla_X Y, Z) + \mu g(Y, \nabla_X Z)$$

$$= \mu g(\nabla_X Y + S(X,Y), Z) + \mu g(Y, \nabla_X Z + S(X,Z)) = \bar{g}(\bar{\nabla}_X Y, Z) + \bar{g}(Y, \bar{\nabla}_X Z).$$

By the uniqueness of Riemannian connections on a Riemannian manifold (Theorem 3.6 in Chapter 2, P55), we know that $\bar{\nabla}$ defined as $\bar{\nabla}_X Y = \nabla_X Y + S(X,Y)$ is the Riemannian connection of $(M, \bar{g})$.

6. (Umbilic hypersurfaces of the hyperbolic space). Let (M^{n+1}, g) be a manifold with a Riemannian metric g and let ∇ be its Riemannian connection. We say an immersion $x: N^n \longrightarrow M^{n+1}$ is (totally) umbilic if for all $p \in N$, the second fundamental form B of x at p satisfies $\langle B(X,Y), \eta \rangle(p) = \lambda(p)\langle X,Y \rangle$, $\lambda(p) \in \mathbb{R}$, for all $X, Y \in \mathcal{X}(N)$ and for a given unit field η normal to $x(N)$; here we are using $\langle\ ,\ \rangle$ to denote the metric g on M and the metric induced by x on N.

a) Show that if M^{n+1} has constant sectional curvature, λ does not depend on p.
Hint: Let $T, X, Y \in \mathcal{X}(N)$. The given condition implies that $-\langle \nabla_X \eta, Y \rangle = \lambda \langle X, Y \rangle$ and $-\langle \nabla_T \eta, Y \rangle = \lambda \langle T, Y \rangle$. Differentiate the first equation with respect to T and the second with respect to X, obtaining, for all Y,

$$\langle \nabla_T \nabla_X \eta - \nabla_X \nabla_T \eta, Y \rangle = -\langle T(\lambda)X - X(\lambda)T + \nabla_{[X,T]}\eta, Y \rangle.$$

Use the fact that M has constant sectional curvature to conclude that $T(\lambda)X - X(\lambda)T = 0$. Because T and X can be chosen independently, this implies that $X(\lambda) = 0$; for all $X \in \mathcal{X}(N)$; therefore $\lambda = \text{const}$.

b) Use Exercise 5 to show that if we change the metric g to a metric $\bar{g} = \mu g$, conformal to g, the immersion $x: N^n \longrightarrow (M^{n+1}, \bar{g})$ continues being umbilic, that is, if (using the notation of Exercise 5) $\langle \nabla_X \eta, Y \rangle_g = -\lambda \langle X, Y \rangle_g$, then

$$\left\langle \bar{\nabla}_X \left(\frac{\eta}{\sqrt{\mu}} \right), Y \right\rangle_{\bar{g}} = \frac{-2\lambda\mu + \eta(\mu)}{2\mu\sqrt{\mu}} \langle X, Y \rangle_{\bar{g}}.$$

c) Take $M^{n+1} = \mathbb{R}^{n+1}$ with the euclidean metric. Show that if $x: N^n \longrightarrow \mathbb{R}^{n+1}$ is umbilic, then $x(N)$ is contained in an n-plane or an n-sphere in $\mathbb{R}^{n+1}$.
Hint: For (a), $\lambda = $ constant. If $\lambda = 0$, $\langle \nabla_X \eta, Y \rangle = 0$ for all $X, Y \in \mathcal{X}(N)$ and all $\eta \in \mathcal{X}(N)^{\perp}$. It follows that $x(N)$ is contained in an affine n-plane in $\mathbb{R}^{n+1}$. If $\lambda \neq 0$, consider the map $y: N \longrightarrow \mathbb{R}^{n+1}$ given by $y(p) = x(p) + \frac{\eta(p)}{\lambda}$, $p \in N$. Let $T, Y \in \mathcal{X}(N)$. Observe that $\langle \nabla_T y, Y \rangle = \langle T, Y \rangle + \frac{1}{\lambda} \langle \nabla_T \eta, Y \rangle = 0$. It follows that $y(N)$ reduces to a point, call it x_0, and that x satisfies $|x(p) - x_0|^2 = 1/\lambda^2$, that is, $x(N)$ is contained in a sphere of center x_0 and radius $1/|\lambda|$.

d) Use (b) and (c) to establish that the umbilic hypersurfaces of the hyperbolic space, in the upper half-space model H^{n+1}, are the intersections with H^{n+1} of n-planes or n-spheres of $\mathbb{R}^{n+1}$. Therefore, the umbilic hypersurfaces of the hyperbolic space are the geodesic spheres, the horospheres and the hyperspheres. Conclude that such

No.

Date

hypersurfaces have constant sectional curvature.

e) Calculate the mean curvature and the sectional curvature of the umbilic hypersurfaces of the hyperbolic space.

Hint: Consider the model of H^n as the upper half-space. Let $\Sigma = S \cap H^n$ be the intersection of H^n with a Euclidean $(n-1)$-sphere $S \subset \mathbb{R}^n$ of radius 1 and center in H^n. Since Σ is umbilic, all of the directions are principal, and it is enough to calculate the curvature of the curves of intersection of Σ with the $x_1 x_n$-plane. Use the expression obtained in part (b) of this exercise to establish that the mean curvature of Σ (in the metric of H^n) is equal to 1 if S is tangent to ∂H^n, is equal to $\cos\alpha$ if S makes an angle α with ∂H^n, and is equal to the "height" of the Euclidean center of S relative to ∂H^n, if $S \subset H^n$. To calculate the sectional curvature, use the Gauss formula.

• $\boxed{a)}$ Since $\dim N = n = (n+1)-1 = \dim M - 1$, we know $\dim (T_pN)^\perp = 1$ for all $p \in N$. Note that η is a unit normal vector field to $x(N)$, thus $\langle B(X,Y), \eta\rangle = \lambda\langle X,Y\rangle$ implies that $B(X,Y) = \lambda\langle X,Y\rangle\eta$ at each $p \in x(N)$. Let $T, X, Y \in \mathfrak{X}(N)$ be arbitrarily chosen.

Then
$$-\langle \nabla_X\eta, Y\rangle = \langle -(\nabla_X\eta)^T, Y\rangle = \langle S_\eta(X), Y\rangle = \langle B(X,Y),\eta\rangle = \lambda\langle X,Y\rangle$$
$$-\langle \nabla_T\eta, Y\rangle = \langle -(\nabla_T\eta)^T, Y\rangle = \langle S_\eta(T), Y\rangle = \langle B(T,Y),\eta\rangle = \lambda\langle T,Y\rangle.$$

Differentiating the first equality with respect to T and the second with respect to X, one obtain
$$-\langle \nabla_T\nabla_X\eta, Y\rangle - \langle \nabla_X\eta, \nabla_TY\rangle = T(\lambda)\langle X,Y\rangle + \lambda\langle \nabla_TX, Y\rangle + \lambda\langle X, \nabla_TY\rangle$$
$$-\langle \nabla_X\nabla_T\eta, Y\rangle - \langle \nabla_T\eta, \nabla_XY\rangle = X(\lambda)\langle T,Y\rangle + \lambda\langle \nabla_XT, Y\rangle + \lambda\langle T, \nabla_XY\rangle$$

Note that
$$-\langle \nabla_X\eta, \nabla_TY\rangle = \langle -(\nabla_X\eta)^T, \nabla_TY\rangle = \langle S_\eta(X), \nabla_TY\rangle = \langle B(X,\nabla_TY),\eta\rangle = \lambda\langle X, \nabla_TY\rangle$$
$$-\langle \nabla_T\eta, \nabla_XY\rangle = \langle -(\nabla_T\eta)^T, \nabla_XY\rangle = \langle S_\eta(T), \nabla_XY\rangle = \langle B(T,\nabla_XY),\eta\rangle = \lambda\langle T, \nabla_XY\rangle$$

therefore
$$-\langle \nabla_T\nabla_X\eta, Y\rangle = T(\lambda)\langle X,Y\rangle + \lambda\langle \nabla_TX, Y\rangle$$
$$-\langle \nabla_X\nabla_T\eta, Y\rangle = X(\lambda)\langle T,Y\rangle + \lambda\langle \nabla_XT, Y\rangle$$

which gives
$$\begin{aligned}\langle \nabla_T\nabla_X\eta - \nabla_X\nabla_T\eta, Y\rangle &= -\langle T(\lambda)X - X(\lambda)T, Y\rangle + \lambda\langle \nabla_XT - \nabla_TX, Y\rangle \\ &= -\langle T(\lambda)X - X(\lambda)T, Y\rangle + \lambda\langle [X,T], Y\rangle \\ &= -\langle T(\lambda)X - X(\lambda)T, Y\rangle + \langle B([X,T],Y),\eta\rangle \\ &= -\langle T(\lambda)X - X(\lambda)T, Y\rangle + \langle -\nabla_{[X,T]}\eta, Y\rangle\end{aligned}$$

or equivalently $\langle R(X,T)\eta, Y\rangle = \langle \nabla_T\nabla_X\eta - \nabla_X\nabla_T\eta + \nabla_{[X,T]}\eta, Y\rangle = -\langle T(\lambda)X - X(\lambda)T, Y\rangle$.

Since M has constant sectional curvature, by Lemma 3.4 in Chapter 4 (P96),

$\langle R(X,T)\eta, Y\rangle = K_0\left(\cancel{\langle X,\eta\rangle}\langle T,Y\rangle - \cancel{\langle T,\eta\rangle}\langle X,Y\rangle\right) = 0$ where K_0 is some constant.

It follows that $T(\lambda)X - X(\lambda)T = 0$ for all $T, X \in \mathfrak{X}(N)$. Since T, X can be chosen linearly independently, this implies $T(\lambda) = X(\lambda) = 0$ for all T, X linearly independently, in particular $X(\lambda) = 0$ for all $X \in \mathfrak{X}(N)$. Thus $\lambda = $ const. does not depend on $p \in N$.

$\boxed{b)}$ Direct computation yields

$$\left\langle \bar{\nabla}_X\left(\frac{\eta}{\sqrt{\mu}}\right), Y \right\rangle_{\bar{g}} = \mu \left\langle \nabla_X\left(\frac{\eta}{\sqrt{\mu}}\right) + S\left(X, \frac{\eta}{\sqrt{\mu}}\right), Y \right\rangle_g$$

$$= \mu \left\langle -\frac{(X\mu)\cdot\eta}{2\mu^{3/2}} + \frac{\nabla_X\eta}{\sqrt{\mu}} + \frac{1}{2\mu}\left[(X\mu)\frac{\eta}{\sqrt{\mu}} + \frac{\eta(\mu)}{\sqrt{\mu}}X - \cancel{\left\langle X, \frac{\eta}{\sqrt{\mu}}\right\rangle_g}\cdot \operatorname{grad}\mu\right], Y \right\rangle$$

$$= \mu \left\langle \frac{\nabla_X\eta}{\sqrt{\mu}} + \frac{\eta(\mu)X}{2\mu\sqrt{\mu}}, Y \right\rangle_g = -\sqrt{\mu}\langle -\nabla_X\eta, Y\rangle_g + \frac{\eta(\mu)}{2\sqrt{\mu}}\langle X, Y\rangle_g$$

$$= -\sqrt{\mu}\langle B(X,Y), \eta\rangle_g + \frac{\eta(\mu)}{2\sqrt{\mu}}\langle X, Y\rangle_g = -\sqrt{\mu}\cdot\lambda\langle X, Y\rangle_g + \frac{\eta(\mu)}{2\sqrt{\mu}}\langle X, Y\rangle_g$$

$$= \frac{-2\lambda\mu + \eta(\mu)}{2\sqrt{\mu}}\langle X, Y\rangle_g = \frac{-2\lambda\mu + \eta(\mu)}{2\mu\sqrt{\mu}}\langle X, Y\rangle_{\bar{g}}.$$

$\boxed{c)}$ By a), λ is constant. If $\lambda = 0$, then $B(X,Y) = \lambda\langle X, Y\rangle\eta = 0$ for all $X, Y \in T_pN$ and all $p \in N$. Thus N is a totally geodesic submanifold of $M^{n+1} = \mathbb{R}^{n+1}$, and hence $x(N)$ is contained in an affine n-plane in $\mathbb{R}^{n+1}$. If $\lambda \neq 0$, consider the mapping $y: N \longrightarrow \mathbb{R}^{n+1}$ given by $y(p) = x(p) + \frac{\eta(p)}{\lambda}$, $p \in N$. Let $T, Y \in \mathfrak{X}(N)$ be arbitrarily chosen, then one has $\langle \nabla_T y, Y\rangle = \left\langle \nabla_T x + \frac{\nabla_T \eta}{\lambda}, Y\right\rangle = \langle T, Y\rangle + \frac{1}{\lambda}\langle \nabla_T\eta, Y\rangle =$
$= \langle T, Y\rangle - \frac{1}{\lambda}\langle -\nabla_T\eta, Y\rangle = \langle T, Y\rangle - \frac{1}{\lambda}\langle S_\eta(T), Y\rangle = \langle T, Y\rangle - \frac{1}{\lambda}\langle B(T,Y), \eta\rangle = \langle T, Y\rangle - \frac{1}{\lambda}\cdot\lambda\langle T, Y\rangle =$
$= \langle T, Y\rangle - \langle T, Y\rangle = 0$. Thus $\nabla_T y \equiv 0$ for all $p \in N$, i.e. $y(p)$ is a constant point in $\mathbb{R}^{n+1}$. Denote $x_0 \equiv y(p)$ for all $p \in N$, then $|x(p) - x_0|^2 = |\eta(p)|^2/\lambda^2 = 1/\lambda^2$, or equivalently speaking $x(N) \subset S^n_{1/|\lambda|}(x_0)$.

$\boxed{d)}$ Denote $g_{ij} = \delta_{ij}$ and $\bar{g}_{ij} = \delta_{ij}/(x^{n+1})^2$. Then $\bar{g} = \frac{1}{(x^{n+1})^2}g$. By b) c) we know the umbilic hypersurfaces in $(H^{n+1}, \bar{g})$ coincides with the umbilic hypersurfaces in (H^{n+1}, g). Hence the umbilic hypersurfaces of $(H^{n+1}, \bar{g})$ are the intersections with H^{n+1} of n-planes or n-spheres of $\mathbb{R}^{n+1}$, because (H^{n+1}, g) is an open submanifold of $(\mathbb{R}^{n+1}, g) = (\mathbb{R}^{n+1}$, Euclidean$)$. Therefore, the umbilic hypersurfaces of the hyperbolic space are the geodesic spheres, the horospheres and the hyperspheres. By the Gauss Formula (Theorem 2.5 in Chapter 6, P130), noting that the ambient space $(H^{n+1}, \bar{g})$ has constant sectional curvature -1, one has $K(X,Y) - (-1) = \langle B(X,X), B(Y,Y)\rangle - |B(X,Y)|^2 = \lambda^2\langle X,X\rangle\langle Y,Y\rangle - \lambda^2\langle X,Y\rangle\langle X,Y\rangle = \lambda^2$ for all

No.

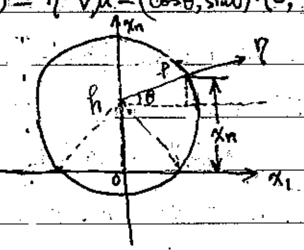
Date

$X, Y \in T_p\Sigma$ orthonormal and all $p \in \Sigma$, where Σ is any of the umbilic hypersurfaces of $(H^{n+1}, \bar{g})$ mentioned above. Hence $K(p,\sigma) \equiv -1 + \lambda^2 = \text{const.}$ on Σ, in other words the umbilic hypersurfaces of the hyperbolic space all have constant sectional curvature.

$\boxed{e)}$ We follow the hint closely. Consider the model of H^n as the upper half-plane $(H^n, \bar{g}_{ij} = \delta_{ij}/(x^n)^2)$. Let $\Sigma = S \cap H^n$ be the intersection of H^n with a Euclidean $(n-1)$-sphere S, on which a curve of intersection of Σ with the x_1x_n-plane is given by

$$(-\varepsilon, \varepsilon) \ni \theta \longmapsto \Gamma(\theta) = (\cos\theta,\ h + \sin\theta)$$

where h is the "height" of the Euclidean center of S relative to ∂H^n. Here S is chosen such that it has radius 1. By b) of this Exercise, we know if under the metric $g_{ij} = \delta_{ij}$ one has $\langle \nabla_X \eta, Y\rangle_g = -\lambda \langle X, Y\rangle_g$ for all $X, Y \in T_p\Sigma$ and all $p \in \Sigma$ then $\left\langle \bar{\nabla}_X\left(\frac{\eta}{\sqrt{\mu}}\right), Y\right\rangle_{\bar{g}} = \frac{-2\lambda\mu + \eta(\mu)}{2\mu\sqrt{\mu}} \langle X, Y\rangle_{\bar{g}}$ where $\frac{\eta}{\sqrt{\mu}}$ is a unit normal vector field on $X(N)$. By the definition of the mean curvature in Chapter 6 (P133), one has $H(p) = -\left.\frac{-2\lambda\mu + \eta(\mu)}{2\mu\sqrt{\mu}}\right|_p$. Hence it suffices to compute λ and μ explicitly. To determine λ, note that in (H^{n+1}, δ_{ij}) one has $\nabla_X \eta = X(\eta) = (X^1\partial_1 + X^n\partial_n)(x^1\partial_1 + x^n\partial_n) = X^1\partial_1 + X^n\partial_n = X$, and hence $\lambda\langle X, Y\rangle = \langle -\nabla_X\eta, Y\rangle = -\langle X, Y\rangle \Rightarrow \lambda = -1$. That $\mu = 1/(x_n)^2$ then gives $\eta(\mu) = \eta \cdot \nabla\mu = (\cos\theta, \sin\theta)\cdot\left(0, -\frac{2}{x_n^3}\right) = -\frac{2\sin\theta}{x_n^3} = -\frac{2(x_n - h)}{x_n^3}$ where we keep $\eta = (\cos\theta, \sin\theta)$.



Hence
$$H(p) = \frac{2\lambda\mu - \eta(\mu)}{2\mu\sqrt{\mu}} = \frac{1}{\sqrt{\mu}} - \frac{\eta(\mu)}{2\mu\sqrt{\mu}}$$
$$= -x_n + \frac{2(x_n - h)}{x_n^3}\cdot\frac{x_n^3}{2} = -x_n + x_n - h = -h.$$

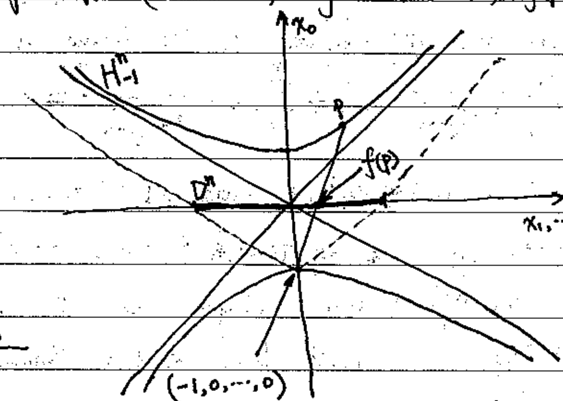
Note that the sign of the mean curvature depends on the choice of normal vector: if we choose $\eta = (-\cos\theta, -\sin\theta)$ then $H(p) = h$. Note also that $h = 1$ if S is tangent to ∂H^n, and $h = \cos\alpha$ if S makes an angle α with ∂H^n, we have verified all statements in the hint. To calculate the sectional curvature, as we have done in the solution to d) by means of the Gauss formula, one obtains $K(p;\sigma) \equiv -1 + h^2$ for all $p \in \Sigma$ and all two dimensional subspaces of $T_p\Sigma$.

7. Define a "stereographic projection" $f: H^n_{-1} \longrightarrow D^n$ from the model of the hyperbolic space H^n_{-1} of curvature -1 given in Exercise 3 onto the open ball

$$D^n = \{(x_0, \cdots, x_n);\ x_0 = 0,\ \sum_{a=1}^n x_a^2 < 1\}$$

in the following way: If $p \in H^n_{-1} \subset L^{n+1}$, join p to $p_0 = (-1, 0, \cdots, 0)$ by a line r; $f(p)$ is the intersection of r with D^n (See Fig. 3).

Let $p = (x_0, \cdots, x_n)$ and $f(p) = (0, u_1, \cdots, u_n)$.



a) Prove that: $x_a = \dfrac{2u_a}{1 - \sum_a u_a^2}$, $a = 1, \cdots, n$,

$$x_0 = \frac{2}{1 - \sum_a u_a^2} - 1.$$

b) Show that:

$$-(dx_0)^2 + (dx_1)^2 + \cdots + (dx_n)^2 = \frac{4[(du_1)^2 + \cdots + (du_n)^2]}{(1 - \sum_a u_a^2)^2}$$

Conclude that $f^{-1}: D^n \longrightarrow H^n_{-1}$ induces on D the metric $g_{ij} = \dfrac{4\delta_{ij}}{(1 - \sum_a u_a^2)^2}$. Therefore, D^n with the metric g_{ij} has constant curvature -1 (cf. Exercise 1(c)).

c) Show that the images by f of the non-empty intersections of affine hyperplanes P of L^{n+1} with H^n_{-1} are intersections with D^n of spheres (or planes, when P passes through p_0) contained in the hyperplane $x_0 = 0$. Conclude that the umbilic hypersurfaces of H^n_{-1} (Cf. Exercise 6) are of the form $P \cap H^n_{-1}$.

• $\boxed{a)}$ Since $p, f(p), p_0$ are colinear, we may write $p - p_0 = \lambda(f(p) - p_0)$ for some $\lambda \in \mathbb{R}$, i.e. $(x_0 + 1, x_1, \cdots, x_n) = \lambda(1, u_1, \cdots, u_n)$. Thus $x_0 = \lambda - 1$ and $x_a = \lambda u_a$, $a = 1, \cdots, n$. By $x_0^2 = \sum_{a=1}^n x_a^2 + 1$ one obtains $\lambda = \dfrac{2}{1 - \sum_a u_a^2}$, hence $x_a = \dfrac{2u_a}{1 - \sum_a u_a^2}$ for $a = 1, \cdots, n$ and $x_0 = \dfrac{2}{1 - \sum_a u_a^2} - 1$.

$\boxed{b)}$ Direct computation yields

$$-(dx_0)^2 + (dx_1)^2 + \cdots + (dx_n)^2 = -\left(2\,d\left(\frac{1}{1 - \sum_a u_a^2}\right)\right)^2 + \sum_{a=1}^n \left[\frac{2\,du_a}{1 - \sum u_a^2} + 2u_a\, d\left(\frac{1}{1 - \sum_a u_a^2}\right)\right]^2$$

$$= -\frac{4 \cdot 4\left(\sum_{a=1}^n u_a du_a\right)^2}{\left(1 - \sum_{a=1}^n u_a^2\right)^4} + \frac{4\sum_{a=1}^n (du_a)^2}{\left(1 - \sum_{a=1}^n u_a^2\right)^2} + \frac{16\left(\sum_{a=1}^n u_a du_a\right)\cdot\left(\sum_{a=1}^n u_a du_a\right)}{\left(1 - \sum_{a=1}^n u_a^2\right)^3} + \frac{4\sum_{a=1}^n u_a^2 \cdot 4\left(\sum_{a=1}^n u_a du_a\right)^2}{\left(1 - \sum_{a=1}^n u_a^2\right)^4}$$

$$\frac{\sigma := \sum_{a=1}^n u_a du_a}{\tau := \sum_{a=1}^n u_a^2} \quad -\frac{16\sigma^2}{(1-\tau)^4} + \frac{16\sigma^2}{(1-\tau)^3} + \frac{16\tau\sigma^2}{(1-\tau)^4} + \frac{4\sum_{a=1}^n (du_a)^2}{\left(1 - \sum_{a=1}^n u_a^2\right)^2}$$

$$= \frac{4\sum_{a=1}^n (du_a)^2}{\left(1 - \sum_{a=1}^n u_a^2\right)^2} - \cancel{\frac{16(1-\tau)\sigma^2}{(1-\tau)^4}} + \cancel{\frac{16\sigma^2}{(1-\tau)^3}} = \frac{4\sum_{a=1}^n (du_a)^2}{\left(1 - \sum_{a=1}^n u_a^2\right)^2} = \frac{4[(du_1)^2 + \cdots + (du_n)^2]}{\left(1 - \sum_a u_a^2\right)^2}$$

No.

Date

Since f is easily verified to be a diffeomorphism, $f^{-1}: D^n \longrightarrow H^n_{-1}$ induces a pullback metric on D^n with formulae $g_{ij} = \delta_{ij}/(1-\sum_a u_a^2)^2$. It follows that (D^n, g_{ij}) is isometric to H^n_{-1}, and hence has constant negative sectional curvature -1.

$\boxed{c)}$ Let $a_0 x_0 + \sum_{i=1}^n a_i x_i = b$ be a hyperplane in L^{n+1} with non-empty intersection with H^n_{-1}. Under the diffeomorphism f, its image in D^n is determined by

$$a_0\left(\frac{2}{1-\sum_a u_a^2} - 1\right) + \sum_{i=1}^n a_i \cdot \frac{2u_i}{1-\sum_a u_a^2} = b \Leftrightarrow 2a_0 + \sum_a 2a_a u_a = (b+a_0)\left(1-\sum_a u_a^2\right)$$

which is a hypersphere contained in the hyperplane $x_0 = 0$. Note that all hyperplanes of this type passing through $P_0 = (-1, 0, \cdots, 0)$ satisfies $-a_0 = b \Leftrightarrow a_0 + b = 0$, and hence the hypersphere above degenerates into a hyperplane contained in the hyperplane $x_0 = 0$; all hyperplanes which does not pass through $P_0 = (-1, 0, \cdots, 0)$ keep $a_0 + b \neq 0$ and hence the hypersphere above is a non-degenerate hypersphere. It is easy to check that the converse is also true: any hypersphere or hyperplane contained in the hyperplane $x_0 = 0$ is the image under f of the non-empty intersection with H^n_{-1} of a hyperplane P of L^{n+1}. By Exercise 6 and the discussion on P178, all umbilic hypersurfaces in D^n are of the form of an intersection with D^n of n-spheres contained in the hyperplane $x_0 = 0$. It follows that the umbilic hypersurfaces of H^n_{-1} are of the form $P \cap H^n_{-1}$ for some hyperplane P of L^{n+1}.

8. (Riemannian submersions). A differentiable mapping $f: \bar{M}^{n+k} \to M^n$ is called submersion if f is surjective, and for all $\bar{p} \in \bar{M}$, $df_{\bar{p}}: T_{\bar{p}}\bar{M} \to T_{f(\bar{p})}M$ has rank n. In this case, for all $p \in M$, the fiber $f^{-1}(p) = F_p$ is a submanifold of $\bar{M}$ and a tangent vector of $\bar{M}$, tangent to some F_p, $p \in M$, is called a vertical vector of the submersion. If, in addition, $\bar{M}$ and M have Riemannian metrics, the submersion f is said to be Riemannian if, for all $p \in \bar{M}$, $df_p: T_p\bar{M} \to T_{f(p)}M$ preserves lengths of vectors orthogonal to F_p. Show that:

a) If $M_1 \times M_2$ is the Riemanian product, then the natural projections $\pi_i: M_1 \times M_2 \to M_i$, $i=1,2$, are Riemannian submersions.

b) If the tangent bundle TM is given the Riemannian metric as in Exercise 2 of Chap. 3, then the projection $\pi: TM \to M$ is a Riemannian submersion.

- $\boxed{a)}$ Denote g_1, g_2 for the Riemanian metrics of M_1, M_2 separately. Then $\pi_i: M_1 \times M_2 \to M_i$, $i=1,2$ are obviously submersions, since π_i's are surjective and $d\pi_{i(m_1,m_2)}: T_{(m_1,m_2)}(M_1 \times M_2) \to T_{m_i}M_i$ has rank $\dim M_i$, both easily verified from their definitions (note that one has $d\pi_1 = (Id, 0)$, $d\pi_2 = (0, Id)$). Let us fix $p = m_1 \in M_1$ arbitrarily, and look at an arbitrary vertical vector $v = (v_1, v_2) \in T_p(M_1 \times M_2)$ orthogonal to $\pi_1^{-1}(p)$. By definition, for any $w = (0, w_2) \in \pi_1^{-1}(p) = \{m_1\} \times M_2$ one has $0 = \langle v, w \rangle_{M_1 \times M_2} = g_1(v_1, w_1) + g_2(v_2, w_2) =$ $= g_2(v_2, w_2)$ and hence $v_2 = 0$. It follows that $\langle v, v \rangle_{M_1 \times M_2} = g_1(v_1, v_1) + g_2(v_2, v_2) = g_1(v_1, v_1) =$ $= \langle (\pi_1)_*(v), (\pi_1)_*(v) \rangle_{M_1}$, which means that $\pi_1: M_1 \times M_2 \to M_1$ is a Riemannian submersion. Similarly, $\pi_2: M_1 \times M_2 \to M_2$ is also a Riemannian submersion.

$\boxed{b)}$ It is easy to check by definition that $\pi: TM \to M$ is a submersion (note that $d\pi_{(p,v)} = (Id_{T_pM}, 0)$ for all $(p,v) \in TM$). Recall from Exercise 2e) of Chapter 3 (on P80) that $\langle W, W \rangle_{(p,v)} = \langle d\pi(W), d\pi(W) \rangle_p$ if W is horizontal and one concludes that the projection $\pi: TM \to M$ is a Riemannian submersion.

No.

Date

9. (Connection of a Riemannian submersion). Let $f: \bar{M} \to M$ be a Riemannian submersion. A vector $\bar{x} \in T_{\bar{p}}\bar{M}$ is <u>horizontal</u> if it is orthogonal to the fiber. The tangent space $T_{\bar{p}}\bar{M}$ then admits a decomposition $T_{\bar{p}}\bar{M} = (T_{\bar{p}}\bar{M})^h \oplus (T_{\bar{p}}\bar{M})^v$, where $(T_{\bar{p}}\bar{M})^h$ and $(T_{\bar{p}}\bar{M})^v$ denote the subspaces of horizontal and vertical vectors, respectively. If $X \in \mathfrak{X}(M)$, the <u>horizontal lift</u> $\bar{X}$ of X is the horizontal field defined by $df_{\bar{p}}(\bar{X}(\bar{p})) = X(f(\bar{p}))$.

a) Show that $\bar{X}$ is differentiable.

b) Let ∇ and $\bar{\nabla}$ be the Riemannian connections of M and $\bar{M}$ respectively. Show that $\bar{\nabla}_{\bar{X}}\bar{Y} = \overline{(\nabla_X Y)} + \frac{1}{2}[\bar{X},\bar{Y}]^v$, $X, Y \in \mathfrak{X}(M)$, where Z^v is the vertical component of Z.

c) $[\bar{X},\bar{Y}]^v(\bar{p})$ depends only on $\bar{X}(\bar{p})$ and $\bar{Y}(\bar{p})$.

Hint for (b): Let $X, Y, Z \in \mathfrak{X}(M)$. Let $T \in \mathfrak{X}(\bar{M})$ be a vertical field. Observe that: $\langle \bar{X}, T\rangle = \langle \bar{Y}, T\rangle = \langle \bar{Z}, T\rangle = 0$, $\bar{X}\langle \bar{Y}, \bar{Z}\rangle = X\langle Y, Z\rangle$, $df[\bar{X}, T] = 0$, $[X,Y] = [df(\bar{X}), df(\bar{Y})] = df[\bar{X},\bar{Y}]$ and $T\langle \bar{X},\bar{Y}\rangle = 0$. Conclude that $\langle [\bar{X},\bar{Y}], \bar{Z}\rangle = \langle [X,Y], Z\rangle$, $\langle [\bar{X},T], \bar{Y}\rangle = 0$ and use the formula for the Riemannian connection as a function of the metric to obtain $\langle \bar{\nabla}_{\bar{X}}\bar{Y}, \bar{Z}\rangle = \langle \nabla_X Y, Z\rangle$, $2\langle \bar{\nabla}_{\bar{X}}\bar{Y}, T\rangle = \langle T, [\bar{X},\bar{Y}]\rangle$, which implies (b).

Hint for (c). Use the fact that $\langle [\bar{X},\bar{Y}], T\rangle = \langle \bar{\nabla}_{\bar{X}}\bar{Y} - \bar{\nabla}_{\bar{Y}}\bar{X}, T\rangle$.

- $\boxed{a)}$ For each $\bar{p} \in \bar{M}$, by the Rank Theorem (cf. John Lee "Introduction to Smooth Manifolds" Theorem 5.13 (P107)), there exist a local coordinate system $(x^1, \cdots, x^m)$ around $\bar{p}$ in $\bar{M}$ and a local coordinate system $(y^1, \cdots, y^n)$ around $p = f(\bar{p}) \in M$ in M such that f has the coordinate representation $f(x^1, \cdots, x^m) = (y^1, \cdots, y^n)$, where $\dim \bar{M} =: m \geq n := \dim M$. For each $X \in \mathfrak{X}(M)$, locally it has coordinate representation $X(y^1, \cdots, y^n) = \sum_{k=1}^{n} X^k \frac{\partial}{\partial y^k}\Big|_{(y^1,\cdots,y^n)}$. Define a local vector field $\tilde{X}$ around $\bar{p}$ as $\tilde{X}(x^1, \cdots, x^m) = \sum_{k=1}^{n} (X^k \circ f)(x^1, \cdots, x^m) \frac{\partial}{\partial x^k}\Big|_{(x^1,\cdots,x^m)}$, under the previously chosen coordinate chart around $\bar{p}$. It is easily checked from the chain rule that $\tilde{X}$ is a well-defined local vector field. Using a partition of unity, one can construct $\tilde{X} \in \mathfrak{X}(\bar{M})$ for a globally defined vector field $X \in \mathfrak{X}(M)$ on M, such that $df_{\bar{q}}(\tilde{X}(\bar{q})) = X(f(\bar{q}))$ for all $\bar{q} \in \bar{M}$ and $f(\bar{q}) = q \in M$. Moreover, note that $df_{\bar{q}}\left(\frac{\partial}{\partial x^j}\Big|_{(x^1,\cdots,x^m)}\right) = 0$ for all $j = n+1, \cdots, m$ around $\bar{p}$ under the previously chosen coordinate chart, thus $\left\{\frac{\partial}{\partial x^j}\Big|_{(x^1,\cdots,x^m)}\right\}_{j=n+1}^{m}$ forms a local frame for the local vertical tangent bundle (in other words, $\left\{\frac{\partial}{\partial x^j}\Big|_{\bar{q}}\right\}_{j=n+1}^{m}$ span a basis for $(T_{\bar{q}}\bar{M})^v$ for all $\bar{q}$ in the previously specified coordinate neighborhood around $\bar{p} \in \bar{M}$). It then follows from a Gram-Schmidt process that we can find a local orthonormal frame $\{E_j\}_{j=n+1}^{m}$ for the local vertical tangent bundle of $\bar{M}$ around $\bar{p} \in \bar{M}$. Define a vector field locally around $\bar{p}$ by $\bar{X}(x^1, \cdots, x^m) = \left(\tilde{X} - \sum_{j=n+1}^{m} \langle \tilde{X}, E_j\rangle_{\bar{M}} E_j\right)(x^1, \cdots, x^m)$.

From our construction, it is obvious that $\bar{X}(\bar{q}) \in (T_{\bar{q}}\bar{M})^h$ for all $\bar{q} \in \bar{M}$ in the neighborhood around $\bar{p}$ specifield above, and $df_{\bar{q}}(\bar{X}(\bar{q})) = df_{\bar{q}}(\tilde{X}(\bar{q})) = X(f(\bar{q}))$ because $df_{\bar{q}}(E_j(\bar{q})) = 0$, $j = n+1, \cdots, m$. Note that at each $\bar{q} \in \bar{M}$ there is a unique vector in $(T_{\bar{q}}\bar{M})^h$ whose image under $df_{\bar{q}}$ is $X(f(\bar{q}))$, thus $\bar{X}$ is the horizontal lift of X and is globally well-defined. Now it is obvious that $\bar{X}$ is differentiable around $\bar{p}$, for $\tilde{X}, \{E_j\}_{j=n+1}^{m}$ are smooth around $\bar{p}$ and the Riemannian metric tensor is smooth on $\bar{M}$. By the arbitrary choices of $\bar{p} \in \bar{M}$, we know that $\bar{X}$ is differentiable.

$\boxed{b)}$ Let $X, Y, Z \in \mathfrak{X}(M)$, and denote $\bar{X}, \bar{Y}, \bar{Z}$ for their horizontal lifts respectively. Let $T \in \mathfrak{X}(\bar{M})$ be a vertical field. By definition, $\langle \bar{X}, T\rangle = \langle \bar{Y}, T\rangle = \langle \bar{Z}, T\rangle = 0$. Since a Riemannian submersion preserves lengths of horizontal vectors, one has $\langle \bar{Y}, \bar{Z}\rangle_{\bar{p}} = \langle Y, Z\rangle_{f(\bar{p})} = \langle df_{\bar{p}}(\bar{Y}(\bar{p})), df_{\bar{p}}(\bar{Z}(\bar{p}))\rangle_{f(\bar{p})}$, and hence $X\langle Y, Z\rangle = df_{\bar{p}}(\bar{X}(\bar{p}))(\langle Y, Z\rangle_p) = \bar{X}(\bar{p})(\langle Y,Z\rangle \circ f)(\bar{p}) = \bar{X}\langle \bar{Y}, \bar{Z}\rangle$. Moreover, $df[\bar{X}, T] = [df(\bar{X}), df(T)] = [X, 0] = 0$, and $[X, Y] = [df(\bar{X}), df(\bar{Y})] = df[\bar{X}, \bar{Y}]$. Also, note that $T\langle \bar{Y}, \bar{Z}\rangle = df_{\bar{p}}(T(\bar{p}))(\langle Y, Z\rangle_p) = 0$ since $df_{\bar{p}}(T(\bar{p})) = 0$. Now we can conclude that $\langle [\bar{X}, T], \bar{Y}\rangle = \langle 0, \bar{Y}\rangle = 0$, and $\langle [X,Y], Z\rangle = \langle df[\bar{X},\bar{Y}], df(\bar{Z})\rangle = \langle [\bar{X},\bar{Y}], \bar{Z}\rangle$. By the Koszul Formula,

$$\langle \bar{\nabla}_{\bar{Y}}\bar{X}, \bar{Z}\rangle = \frac{1}{2}\left[\bar{X}\langle \bar{Y},\bar{Z}\rangle + \bar{Y}\langle \bar{Z},\bar{X}\rangle - \bar{Z}\langle \bar{X},\bar{Y}\rangle - \langle[\bar{X},\bar{Z}],\bar{Y}\rangle - \langle[\bar{Y},\bar{Z}],\bar{X}\rangle - \langle[\bar{X},\bar{Y}],\bar{Z}\rangle\right]$$
$$= \frac{1}{2}\left[X\langle Y,Z\rangle + Y\langle Z,X\rangle - Z\langle X,Y\rangle - \langle[X,Z],Y\rangle - \langle[Y,Z],X\rangle - \langle[X,Y],Z\rangle\right] = \langle \nabla_Y X, Z\rangle$$
$$\langle \bar{\nabla}_{\bar{Y}}\bar{X}, T\rangle = \frac{1}{2}\left[\cancel{\bar{X}\langle \bar{Y},T\rangle} + \cancel{\bar{Y}\langle T,\bar{X}\rangle} - \cancel{T\langle \bar{X},\bar{Y}\rangle} - \cancel{\langle[\bar{X},T],\bar{Y}\rangle} - \cancel{\langle[\bar{Y},T],\bar{X}\rangle} - \langle[\bar{X},\bar{Y}],T\rangle\right]$$
$$= \frac{1}{2}\langle[\bar{Y},\bar{X}],T\rangle = \left\langle \frac{1}{2}[\bar{Y},\bar{X}]^v, T\right\rangle$$

By the arbitrariness of Z and T, and the orthogonality of horizontal and vertical subbundles of the tangent bundle, we conclude that (noting that $\langle \nabla_Y X, Z\rangle = \langle \overline{\nabla_Y X}, \bar{Z}\rangle$)

$$\bar{\nabla}_{\bar{Y}}\bar{X} = \overline{\nabla_Y X} + \frac{1}{2}[\bar{Y},\bar{X}]^v \quad \text{for all } X, Y \in \mathfrak{X}(M)$$

or

$$\bar{\nabla}_{\bar{X}}\bar{Y} = \overline{\nabla_X Y} + \frac{1}{2}[\bar{X},\bar{Y}]^v \quad \text{for all } X, Y \in \mathfrak{X}(M).$$

$\boxed{c)}$ Note that for any $T \in \mathfrak{X}(\bar{M})$ a vertical vector field one has

$$\langle [\bar{X},\bar{Y}]^v, T\rangle = \langle [\bar{X},\bar{Y}], T\rangle = \langle \bar{\nabla}_{\bar{X}}\bar{Y} - \bar{\nabla}_{\bar{Y}}\bar{X}, T\rangle$$

thus $[\bar{X},\bar{Y}]^v$ depends only on $\bar{X}$ and $\bar{Y}$, but not X, Y as it appears in the formula we obtained in b). Even better, $[\bar{X},\bar{Y}]^v$ is tensorial in both inputs, since for any $f \in C^\infty(\bar{M})$ one has $\langle [\bar{X}, f\bar{Y}]^v, T\rangle = \langle [\bar{X}, f\bar{Y}], T\rangle = \langle \bar{\nabla}_{\bar{X}}(f\bar{Y}) - \bar{\nabla}_{f\bar{Y}}\bar{X}, T\rangle =$
$= \langle f\bar{\nabla}_{\bar{X}}\bar{Y} + \bar{X}(f)\bar{Y} - f\bar{\nabla}_{\bar{Y}}\bar{X}, T\rangle = \langle f\bar{\nabla}_{\bar{X}}\bar{Y} - f\bar{\nabla}_{\bar{Y}}\bar{X}, T\rangle = \langle f[\bar{X},\bar{Y}], T\rangle = \langle f[\bar{X},\bar{Y}]^v, T\rangle$
which means $[\bar{X}, f\bar{Y}]^v = f[\bar{X},\bar{Y}]^v$. The tensorial property on the other input follows from the skew symmetry of $[\bar{X},\bar{Y}]^v$ (which again follows from $\langle [\bar{X},\bar{Y}]^v, T\rangle = \langle \bar{\nabla}_{\bar{X}}\bar{Y} - \bar{\nabla}_{\bar{Y}}\bar{X}, T\rangle$ for arbitrary vertical $T \in \mathfrak{X}(\bar{M})$).

No.

Date

10. (Curvature of a Riemannian submersion). Let $f: \bar{M} \to M$ be a Riemannian submersion. Let $X, Y, Z, W \in \mathfrak{X}(M)$, $\bar{X}, \bar{Y}, \bar{Z}, \bar{W}$ be their horizontal lifts, and let R and $\bar{R}$ be the curvature tensors of M and $\bar{M}$ respectively. Prove that:

a) $\langle \bar{R}(\bar{X},\bar{Y})\bar{Z}, \bar{W} \rangle = \langle R(X,Y)Z, W \rangle - \frac{1}{4}\langle [\bar{X},\bar{Z}]^v, [\bar{Y},\bar{W}]^v \rangle + \frac{1}{4}\langle [\bar{Y},\bar{Z}]^v, [\bar{X},\bar{W}]^v \rangle$
$- \frac{1}{2}\langle [\bar{Z},\bar{W}]^v, [\bar{X},\bar{Y}]^v \rangle$.

b) $K(\sigma) = \bar{K}(\bar{\sigma}) + \frac{3}{4}\left|[\bar{X},\bar{Y}]^v\right|^2 \geq \bar{K}(\bar{\sigma})$, where σ is the plane generated by the orthonormal vectors $X, Y \in \mathfrak{X}(M)$ and $\bar{\sigma}$ is the plane generated by $\bar{X}, \bar{Y}$.

Hint for (a): We shall use the notation of Exercise 9. Observe that $\bar{X}\langle \bar{\nabla}_{\bar{Y}}\bar{Z}, \bar{W}\rangle = X\langle \nabla_Y Z, W\rangle$. Therefore $\langle \bar{\nabla}_{\bar{X}}\bar{\nabla}_{\bar{Y}}\bar{Z}, \bar{W}\rangle = \bar{X}\langle \bar{\nabla}_{\bar{Y}}\bar{Z}, \bar{W}\rangle - \langle \bar{\nabla}_{\bar{Y}}\bar{Z}, \bar{\nabla}_{\bar{X}}\bar{W}\rangle$
$= \langle \nabla_X\nabla_Y Z, W\rangle - \frac{1}{4}\langle [\bar{Y},\bar{Z}]^v, [\bar{X},\bar{W}]^v\rangle$.

On the other hand, if $T \in \mathfrak{X}(\bar{M})$ is vertical, $\langle \bar{\nabla}_T \bar{X}, \bar{Y}\rangle = \langle \bar{\nabla}_{\bar{X}} T, \bar{Y}\rangle + \langle [T,\bar{X}], \bar{Y}\rangle = -\langle T, \bar{\nabla}_{\bar{X}}\bar{Y}\rangle$.
Therefore, $\langle \bar{\nabla}_{[\bar{X},\bar{Y}]}\bar{Z}, \bar{W}\rangle = \langle \bar{\nabla}_{[\bar{X},\bar{Y}]^h}\bar{Z}, \bar{W}\rangle + \langle \bar{\nabla}_{[\bar{X},\bar{Y}]^v}\bar{Z}, \bar{W}\rangle = \langle \nabla_{[X,Y]}Z, W\rangle - \frac{1}{2}\langle [\bar{X},\bar{Y}]^v, [\bar{Z},\bar{W}]^v\rangle$.
Putting the above together, we obtain (a).

- $\boxed{a)}$ We proceed as the hint suggests. Note that in Exercise 9 we have shown that $\langle \nabla_Y Z, W\rangle = \langle \overline{\nabla_Y Z}, \bar{W}\rangle$ and thus $X\langle \nabla_Y Z, W\rangle = \bar{X}\langle \overline{\nabla_Y Z}, \bar{W}\rangle = \bar{X}\langle \bar{\nabla}_{\bar{Y}}\bar{Z} - \frac{1}{2}[\bar{Y},\bar{Z}]^v, \bar{W}\rangle =$
$= \bar{X}\langle \bar{\nabla}_{\bar{Y}}\bar{Z}, \bar{W}\rangle$. Then $\langle \bar{\nabla}_{\bar{X}}\bar{\nabla}_{\bar{Y}}\bar{Z}, \bar{W}\rangle = \bar{X}\langle \bar{\nabla}_{\bar{Y}}\bar{Z}, \bar{W}\rangle - \langle \bar{\nabla}_{\bar{Y}}\bar{Z}, \bar{\nabla}_{\bar{X}}\bar{W}\rangle = X\langle \nabla_Y Z, W\rangle - \langle \overline{\nabla_Y Z} + \frac{1}{2}[\bar{Y},\bar{Z}]^v,$
$\overline{\nabla_X W} + \frac{1}{2}[\bar{X},\bar{W}]^v\rangle = \langle \nabla_X\nabla_Y Z, W\rangle + \cancel{\langle \nabla_Y Z, \nabla_X W\rangle} - \cancel{\langle \overline{\nabla_Y Z}, \overline{\nabla_X W}\rangle} - \frac{1}{4}\langle [\bar{Y},\bar{Z}]^v, [\bar{X},\bar{W}]^v\rangle =$
$= \langle \nabla_X\nabla_Y Z, W\rangle - \frac{1}{4}\langle [\bar{Y},\bar{Z}]^v, [\bar{X},\bar{W}]^v\rangle$. Similarly, $\langle \bar{\nabla}_{\bar{Y}}\bar{\nabla}_{\bar{X}}\bar{Z}, \bar{W}\rangle = \langle \nabla_Y\nabla_X Z, W\rangle - \frac{1}{4}\langle [\bar{X},\bar{Z}]^v, [\bar{Y},\bar{W}]^v\rangle$.
On the other hand, if $T \in \mathfrak{X}(\bar{M})$ is vertical, then $\langle \bar{\nabla}_T\bar{X}, \bar{Y}\rangle = \langle [T,\bar{X}] + \bar{\nabla}_{\bar{X}}T, \bar{Y}\rangle = \langle [T,\bar{X}], \bar{Y}\rangle +$
$+ \langle \bar{\nabla}_{\bar{X}}T, \bar{Y}\rangle = 0 + \langle \bar{\nabla}_{\bar{X}}T, \bar{Y}\rangle = \langle \bar{\nabla}_{\bar{X}}T, \bar{Y}\rangle = \bar{X}\cancel{\langle T, \bar{Y}\rangle} - \langle T, \bar{\nabla}_{\bar{X}}\bar{Y}\rangle = -\langle T, \bar{\nabla}_{\bar{X}}\bar{Y}\rangle$. It follows that $\langle \bar{\nabla}_{[\bar{X},\bar{Y}]}\bar{Z}, \bar{W}\rangle = \langle \bar{\nabla}_{[\bar{X},\bar{Y}]^h}\bar{Z}, \bar{W}\rangle + \langle \bar{\nabla}_{[\bar{X},\bar{Y}]^v}\bar{Z}, \bar{W}\rangle = \langle \nabla_{[X,Y]}Z, W\rangle - \frac{1}{2}\langle [\bar{X},\bar{Y}]^v, [\bar{Z},\bar{W}]^v\rangle$
where we used $df([\bar{X},\bar{Y}]^v) = df([\bar{X},\bar{Y}]) = [X,Y]$ and $\langle \bar{\nabla}_{[\bar{X},\bar{Y}]^v}\bar{Z}, \bar{W}\rangle = -\langle [\bar{X},\bar{Y}]^v, \bar{\nabla}_{\bar{Z}}\bar{W}\rangle =$
$= -\langle [\bar{X},\bar{Y}]^v, \overline{\nabla_Z W} + \frac{1}{2}[\bar{Z},\bar{W}]^v\rangle = -\frac{1}{2}\langle [\bar{X},\bar{Y}]^v, [\bar{Z},\bar{W}]^v\rangle$. Putting everything together, one obtain
$\langle \bar{R}(\bar{X},\bar{Y})\bar{Z}, \bar{W}\rangle = \langle \bar{\nabla}_{\bar{Y}}\bar{\nabla}_{\bar{X}}\bar{Z}, \bar{W}\rangle - \langle \bar{\nabla}_{\bar{X}}\bar{\nabla}_{\bar{Y}}\bar{Z}, \bar{W}\rangle + \langle \bar{\nabla}_{[\bar{X},\bar{Y}]}\bar{Z}, \bar{W}\rangle = \langle \nabla_Y\nabla_X Z, W\rangle - \frac{1}{4}\langle [\bar{X},\bar{Z}]^v, [\bar{Y},\bar{W}]^v\rangle$
$- \langle \nabla_X\nabla_Y Z, W\rangle + \frac{1}{4}\langle [\bar{Y},\bar{Z}]^v, [\bar{X},\bar{W}]^v\rangle + \langle \nabla_{[X,Y]}Z, W\rangle - \frac{1}{2}\langle [\bar{X},\bar{Y}]^v, [\bar{Z},\bar{W}]^v\rangle = \langle R(X,Y)Z, W\rangle - \frac{1}{4}\langle [\bar{X},\bar{Z}]^v, [\bar{Y},\bar{W}]^v\rangle$
$+ \frac{1}{4}\langle [\bar{Y},\bar{Z}]^v, [\bar{X},\bar{W}]^v\rangle - \frac{1}{2}\langle [\bar{Z},\bar{W}]^v, [\bar{X},\bar{Y}]^v\rangle$, as desired.

$\boxed{b)}$ Letting $\bar{Z} = \bar{X}$, $\bar{W} = \bar{Y}$ in the formula we obtained in a), one obtains (using the orthonormal assumption):
$\bar{K}(\bar{\sigma}) = \langle \bar{R}(\bar{X},\bar{Y})\bar{X}, \bar{Y}\rangle = \langle R(X,Y)X, Y\rangle + \frac{1}{4}\langle [\bar{Y},\bar{X}]^v, [\bar{X},\bar{Y}]^v\rangle - \frac{1}{2}\langle [\bar{X},\bar{Y}]^v, [\bar{X},\bar{Y}]^v\rangle = K(\sigma) - \frac{3}{4}\left|[\bar{X},\bar{Y}]^v\right|^2$,
or equivalently that $K(\sigma) = \bar{K}(\bar{\sigma}) + \frac{3}{4}\left|[\bar{X},\bar{Y}]^v\right|^2 \geq \bar{K}(\bar{\sigma})$. Here σ is the plane generated by the orthonormal vectors $X, Y \in \mathfrak{X}(M)$ and $\bar{\sigma}$ is the plane generated by $\bar{X}, \bar{Y}$ (the linear independence of $\bar{X}, \bar{Y}$ follows from the linear independence of X and the fact that $df_{\bar{p}}: (T_{\bar{p}}\bar{M})^v \to T_{f(\bar{p})}M$ is a linear isometry (being surjective by submersion and injective by the length-preserving property)).

11. (The complex projective space). Let $\mathbb{C}^{n+1}\setminus\{0\} = \{(z_0,\cdots,z_n) = Z \neq 0,\ z_j = x_j + iy_j,\ j=0,\cdots,n\}$ be the set of all non-zero $(n+1)$-tuples of complex numbers z_j. Define an equivalence relation on $\mathbb{C}^{n+1}\setminus\{0\}$: $Z=(z_0,\cdots,z_n) \sim W=(w_0,\cdots,w_n)$ if $z_j = \lambda w_j$, $\lambda\in\mathbb{C}$, $\lambda\neq 0$. The equivalence class of Z will be denoted by $[Z]$ (= the complex line passing through the origin and through Z). The set of such classes is called, by analogy with the real case, the complex projective space $\mathbb{C}P^n$ of complex dimension n.

a) Show that $\mathbb{C}P^n$ has a differentiable structure of a manifold of real dimension $2n$ and that $\mathbb{C}P^1$ is diffeomorphic to S^2.

b) Let $(Z,W) = z_0\bar{w}_0 + \cdots + z_n\bar{w}_n$ be the hermitian product on $\mathbb{C}^{n+1}$, where the bar denotes complex conjugation. Identify $\mathbb{C}^{n+1}\approx\mathbb{R}^{2n+2}$ by putting $z_j = x_j+iy_j = (x_j, y_j)$. Show that $S^{2n+1} = \{N\in\mathbb{C}^{n+1}\approx\mathbb{R}^{2n+2};\ (N,N)=1\}$ is the unit sphere in $\mathbb{R}^{2n+2}$.

c) Show that the equivalence relation $\sim$ induces on S^{2n+1} the following equivalence relation: $Z\sim W$ if $e^{i\theta}Z = W$. Establish that there exists a differentiable map (the Hopf fibering) $f: S^{2n+1}\longrightarrow\mathbb{C}P^n$ such that $f^{-1}([Z]) = \{e^{i\theta}N\in S^{2n+1},\ N\in[Z]\cap S^{2n+1},\ 0\leq\theta\leq 2\pi\} = [Z]\cap S^{2n+1}$.

d) Show that f is a submersion.

- $\boxed{a)}$ Let $\mathcal{U}_j := \{[z_0,\cdots,z_n]\in\mathbb{C}P^n;\ z_j\neq 0\}$ for all $0\leq j\leq n$, and define coordinate maps

$$\bar{x}_j : \mathbb{R}^{2n} \longrightarrow \mathcal{U}_j,\quad 0\leq j\leq n$$
$$(x_1,y_1,\cdots,x_n,y_n) \longmapsto [x_1+iy_1,\cdots,x_{j-1}+iy_{j-1},\ 1,\ x_j+iy_j,\cdots,x_n+iy_n]$$

We are going to show that the family $\{(\mathbb{R}^{2n},\bar{x}_j)\}_{j=0}^n$ is a differentiable structure on $\mathbb{C}P^n$. Indeed, any mapping $\bar{x}_j$ is clearly bijective while $\bigcup_{j=0}^n \bar{x}_j(\mathbb{R}^{2n}) = \bigcup_{j=0}^n \mathcal{U}_j = \mathbb{C}P^n$. It remains to show that $\bar{x}_j^{-1}(\mathcal{U}_j\cap\mathcal{U}_k)$ is open in $\mathbb{R}^{2n}$ and that $\bar{x}_k^{-1}\circ\bar{x}_j$, $k=0,\cdots,n$, is differentiable there. Now, if $j\neq k$, $\bar{x}_j^{-1}(\mathcal{U}_j\cap\mathcal{U}_k) = \{(x_1,y_1,\cdots,x_n,y_n)\in\mathbb{R}^{2n};\ x_k^2+y_k^2\neq 0\}$, which is open in $\mathbb{R}^{2n}$; if $j=k$, $\bar{x}_j^{-1}(\mathcal{U}_j\cap\mathcal{U}_k) = \mathbb{R}^{2n}$, which is also open in $\mathbb{R}^{2n}$. For $j>k$ (the case $j<k$ is similar),

$$(\bar{x}_k^{-1}\circ\bar{x}_j)(x_1,y_1,\cdots,x_n,y_n) = \bar{x}_k^{-1}([x_1+iy_1,\cdots,x_{j-1}+iy_{j-1},1,x_j+iy_j,\cdots,x_n+iy_n]) =$$

$$= \bar{x}_k^{-1}\Bigg(\Bigg[\frac{(x_1x_k+y_1y_k)+i(x_ky_1-x_1y_k)}{x_k^2+y_k^2},\cdots,\frac{(x_{k-1}x_k+y_{k-1}y_k)+i(x_ky_{k-1}-x_{k-1}y_k)}{x_k^2+y_k^2},\ 1,\ \frac{(x_{k+1}x_k+y_{k+1}y_k)+i(x_ky_{k+1}-x_{k+1}y_k)}{x_k^2+y_k^2},$$
$$\cdots,\frac{(x_{j-1}x_k+y_{j-1}y_k)+i(x_ky_{j-1}-x_{j-1}y_k)}{x_k^2+y_k^2},\ \frac{x_k-iy_k}{x_k^2+y_k^2},\ \frac{(x_{j+1}x_k+y_{j+1}y_k)+i(x_ky_{j+1}-x_{j+1}y_k)}{x_k^2+y_k^2},\cdots,\frac{(x_nx_k+y_ny_k)+i(x_ky_n-x_ny_k)}{x_k^2+y_k^2}\Bigg]\Bigg)$$

$$= \Bigg(\frac{x_1x_k+y_1y_k}{x_k^2+y_k^2},\frac{x_ky_1-x_1y_k}{x_k^2+y_k^2},\cdots,\frac{x_{k-1}x_k+y_{k-1}y_k}{x_k^2+y_k^2},\frac{x_ky_{k-1}-x_{k-1}y_k}{x_k^2+y_k^2},\frac{x_{k+1}x_k+y_{k+1}y_k}{x_k^2+y_k^2},\frac{x_ky_{k+1}-x_{k+1}y_k}{x_k^2+y_k^2},\cdots,$$
$$\frac{x_{j-1}x_k+y_{j-1}y_k}{x_k^2+y_k^2},\frac{x_ky_{j-1}-x_{j-1}y_k}{x_k^2+y_k^2},\frac{x_k}{x_k^2+y_k^2},-\frac{y_k}{x_k^2+y_k^2},\frac{x_{j+1}x_k+y_{j+1}y_k}{x_k^2+y_k^2},\frac{x_ky_{j+1}-x_{j+1}y_k}{x_k^2+y_k^2},\cdots,\frac{x_nx_k+y_ny_k}{x_k^2+y_k^2},\frac{x_ky_n-x_ny_k}{x_k^2+y_k^2}\Bigg)$$

No.

Date

which is clearly differentiable. Hence $\mathbb{CP}^n$ has a differentiable structure of a manifold of real dimension $2n$.

To see that $\mathbb{CP}^1$ is diffeomorphic to S^2, recall that $\mathbb{CP}^1 = \mathcal{U}_0 \cup \mathcal{U}_1$ where $\mathcal{U}_0 = \{[z_0, z_1] \in \mathbb{CP}^1 ;\ z_0 \neq 0\}$ and $\mathcal{U}_1 = \{[z_0, z_1] \in \mathbb{CP}^1 ;\ z_1 \neq 0\}$. The coordinate maps are given by $\bar{x}_0 : \mathbb{R}^2 \longrightarrow \mathcal{U}_0 : (x_1, y_1) \longmapsto [1, x_1 + iy_1]$ and $\bar{x}_1 : \mathbb{R}^2 \longrightarrow \mathcal{U}_1 : (x_1, y_1) \longmapsto [x_1 - iy_1, 1]$. The sphere $S^2 = V_0 \cup V_1$ where $V_0 = \{(x_0, x_1, x_2) \in S^2 ;\ x_2 \neq 1\}$ and $V_1 = \{(x_0, x_1, x_2) \in S^2 ;\ x_2 \neq -1\}$, with coordinate maps given by

$$\bar{y}_0 : \mathbb{R}^2 \longrightarrow V_0 : (x, y) \longmapsto \left(\frac{2x}{1+x^2+y^2}, \frac{2y}{1+x^2+y^2}, \frac{-1+x^2+y^2}{1+x^2+y^2} \right)$$

and

$$\bar{y}_1 : \mathbb{R}^2 \longrightarrow V_1 : (x, y) \longmapsto \left(\frac{2x}{1+x^2+y^2}, \frac{2y}{1+x^2+y^2}, \frac{1-x^2-y^2}{1+x^2+y^2} \right).$$

Define a map $h : \mathbb{CP}^1 \longrightarrow S^2$ by

$$h([z_0, z_1]) = h([x_0 + iy_0, x_1 + iy_1]) = \begin{cases} (\bar{y}_0 \circ \bar{x}_0^{-1})([z_0, z_1]) & \text{if } [z_0, z_1] \in \mathcal{U}_0 \\ (\bar{y}_1 \circ \bar{x}_1^{-1})([z_0, z_1]) & \text{if } [z_0, z_1] \in \mathcal{U}_1 \end{cases}$$

$$= \begin{cases} \left(\dfrac{2(x_0x_1 + y_0y_1)}{x_0^2 + y_0^2 + x_1^2 + y_1^2}, \dfrac{2(x_0y_1 - x_1y_0)}{x_0^2 + y_0^2 + x_1^2 + y_1^2}, \dfrac{-(x_0^2 + y_0^2) + (x_1^2 + y_1^2)}{x_0^2 + y_0^2 + x_1^2 + y_1^2} \right) & \text{if } x_0^2 + y_0^2 \neq 0 \\ \left(\dfrac{2(x_0x_1 + y_0y_1)}{x_0^2 + y_0^2 + x_1^2 + y_1^2}, \dfrac{-2(x_1y_0 - x_0y_1)}{x_0^2 + y_0^2 + x_1^2 + y_1^2}, \dfrac{(x_1^2 + y_1^2) - (x_0^2 + y_0^2)}{x_0^2 + y_0^2 + x_1^2 + y_1^2} \right) & \text{if } x_1^2 + y_1^2 \neq 0 \end{cases}$$

$$= \left(\frac{2(x_0x_1 + y_0y_1)}{x_0^2 + y_0^2 + x_1^2 + y_1^2}, \frac{2(x_0y_1 - x_1y_0)}{x_0^2 + y_0^2 + x_1^2 + y_1^2}, \frac{-x_0^2 - y_0^2 + x_1^2 + y_1^2}{x_0^2 + y_0^2 + x_1^2 + y_1^2} \right) \in S^2$$

From the last expression, it is obvious to see that $h : \mathbb{CP}^1 \longrightarrow S^2$ is well-defined on $\mathcal{U}_0 \cap \mathcal{U}_1$. To see that $h : \mathbb{CP}^1 \longrightarrow S^2$ is smooth, it suffices to note that $\bar{y}_0^{-1} \circ h \circ \bar{x}_0 : \mathbb{R}^2 \longrightarrow \mathbb{R}^2$ and $\bar{y}_1^{-1} \circ h \circ \bar{x}_1 : \mathbb{R}^2 \longrightarrow \mathbb{R}^2$ are both identity maps on $\mathbb{R}^2$, and that $\bar{x}_1^{-1} \circ \bar{x}_0 : \mathbb{R}^2 \longrightarrow \mathbb{R}^2$ and $\bar{y}_1^{-1} \circ \bar{y}_0 : \mathbb{R}^2 \longrightarrow \mathbb{R}^2$ are both diffeomorphisms on $\mathbb{R}^2$. Noting that the identity map is invertible with the inverse being smooth (the inverse of an identity map is the identity map itself), we know that $h : \mathbb{CP}^1 \longrightarrow S^1$ is actually a diffeomorphism. This proves that $\mathbb{CP}^1$ is diffeomorphic to S^2.

$\boxed{\text{b)}}$ For $N = (z_0, \cdots, z_n) = (x_0 + iy_0, \cdots, x_n + iy_n) \in \mathbb{C}^{n+1} \approx \mathbb{R}^{2n+2}$, one has $(N, N) = 1 \Leftrightarrow \sum_{k=0}^{n} |z_k|^2 = 1 \Leftrightarrow \sum_{k=0}^{n} (x_k^2 + y_k^2) = 1$, thus the set $\{N \in \mathbb{C}^{n+1} \approx \mathbb{R}^{2n+2} ;\ (N, N) = 1\}$ is identified with the unit sphere S^{2n+1} in $\mathbb{R}^{2n+2}$ with the same identity map which identifies $\mathbb{C}^{n+1}$ with $\mathbb{R}^{2n+2}$.

c) For $Z, W \in S^{2n+1}$, $|Z|^2 = 1 = |W|^2$. If $e^{i\theta}Z = W$ for some $\theta \in \mathbb{R}$, obviously $Z \sim W$ as points in $\mathbb{C}^{n+1}$. Conversely, if $Z \sim W$ as points in $\mathbb{C}^{n+1}$, then $Z = \lambda W$ for some $\lambda \in \mathbb{C}$, and hence $1 = |Z|^2 = |\lambda W|^2 = |\lambda|^2 \cdot |W|^2 = |\lambda|^2 \Rightarrow \lambda = e^{i\theta}$ for some $\theta \in \mathbb{R}$.
Define a map $f: S^{2n+1} \longrightarrow \mathbb{CP}^n$ by $(z_0, \cdots, z_n) \xmapsto{f} [z_0, \cdots, z_n]$. To see that f is differentiable, let us first specify a differentiable structure on S^{2n+1} as follows.
Let $\mathcal{U}_j^1 := \{(x_0, y_0, \cdots, \widehat{x_j}, y_j, \cdots, x_n, y_n) \in \mathbb{R}^{2n+1} ; \sum_{\substack{\ell=1 \\ \ell \neq j}}^n x_\ell^2 + \sum_{\ell=1}^n y_\ell^2 < 1\}$

$\mathcal{U}_j^2 := \{(x_0, y_0, \cdots, x_j, \widehat{y_j}, \cdots, x_n, y_n) \in \mathbb{R}^{2n+1} ; \sum_{\ell=1}^n x_\ell^2 + \sum_{\substack{\ell=1 \\ \ell \neq j}}^n y_\ell^2 < 1\}$

and define
$$\varphi_j^+ : \mathcal{U}_j^1 \longrightarrow S^{2n+1} : (x_0, y_0, \cdots, \widehat{x_j}, y_j, \cdots, x_n, y_n) \longmapsto (x_0, y_0, \cdots, D_j, y_j, \cdots, x_n, y_n)$$
$$\varphi_j^- : \mathcal{U}_j^1 \longrightarrow S^{2n+1} : (x_0, y_0, \cdots, \widehat{x_j}, y_j, \cdots, x_n, y_n) \longmapsto (x_0, y_0, \cdots, -D_j, y_j, \cdots, x_n, y_n)$$
$$\psi_j^+ : \mathcal{U}_j^2 \longrightarrow S^{2n+1} : (x_0, y_0, \cdots, x_j, \widehat{y_j}, \cdots, x_n, y_n) \longmapsto (x_0, y_0, \cdots, x_j, E_j, \cdots, x_n, y_n)$$
$$\psi_j^- : \mathcal{U}_j^2 \longrightarrow S^{2n+1} : (x_0, y_0, \cdots, x_j, \widehat{y_j}, \cdots, x_n, y_n) \longmapsto (x_0, y_0, \cdots, x_j, -E_j, \cdots, x_n, y_n)$$
where $D_j := \sqrt{1 - \sum_{\substack{\ell=0 \\ \ell \neq j}}^n x_\ell^2 - \sum_{\ell=0}^n y_\ell^2}$, $E_j := \sqrt{1 - \sum_{\ell=0}^n x_\ell^2 - \sum_{\substack{\ell=0 \\ \ell \neq j}}^n y_\ell^2}$. By an argument similar to Example 4.7 of Chapter 0 (P20-21), we know that $\{(\mathcal{U}_j^1, \varphi_j^+), (\mathcal{U}_j^1, \varphi_j^-), (\mathcal{U}_j^2, \psi_j^+), (\mathcal{U}_j^2, \psi_j^-)\}_{j=0}^n$ is a differentiable structure on S^n. Now we can directly verify that $f: S^{2n+1} \longrightarrow \mathbb{CP}^n$ is differentiable by noting that all the compositions below are differentiable maps from $\mathbb{R}^{2n+1}$ to $\mathbb{R}^{2n}$ (here $0 \le j \le n$ and $0 \le k \le n$):

$$(\bar{x}_k^{-1} \circ f \circ \varphi_j^+)(x_0, y_0, \cdots, \widehat{x_j}, y_j, \cdots, x_n, y_n) = \bar{x}_k^{-1}([x_0 + iy_0, \cdots, x_{j-1} + iy_{j-1}, D_j + iy_j, x_{j+1} + iy_{j+1}, \cdots, x_n + iy_n])$$

$$= \begin{cases} \Big(\frac{x_0 x_k + y_0 y_k}{x_k^2 + y_k^2}, \frac{x_k y_0 - x_0 y_k}{x_k^2 + y_k^2}, \cdots, \frac{x_{k-1} x_k + y_{k-1} y_k}{x_k^2 + y_k^2}, \frac{x_k y_{k-1} - x_{k-1} y_k}{x_k^2 + y_k^2}, \frac{x_{k+1} x_k + y_{k+1} y_k}{x_k^2 + y_k^2}, \frac{x_k y_{k+1} - x_{k+1} y_k}{x_k^2 + y_k^2}, \cdots, \\ \frac{x_{j-1} x_k + y_{j-1} y_k}{x_k^2 + y_k^2}, \frac{x_k y_{j-1} - x_{j-1} y_k}{x_k^2 + y_k^2}, \frac{D_j x_k + y_j y_k}{x_k^2 + y_k^2}, \frac{x_k y_j - D_j y_k}{x_k^2 + y_k^2}, \frac{x_{j+1} x_k + y_{j+1} y_k}{x_k^2 + y_k^2}, \frac{x_k y_{j+1} - x_{j+1} y_k}{x_k^2 + y_k^2}, \cdots, \frac{x_n x_k + y_n y_k}{x_k^2 + y_k^2}, \\ \frac{x_k y_n - x_n y_k}{x_k^2 + y_k^2} \Big) & \text{if } k < j \\ & (k > j \text{ similar}) \\ \Big(\frac{x_0 D_j + y_0 y_j}{D_j^2 + y_j^2}, \frac{D_j y_0 - x_0 y_j}{D_j^2 + y_j^2}, \cdots, \frac{x_{j-1} D_j + y_{j-1} y_j}{D_j^2 + y_j^2}, \frac{D_j y_{j-1} - x_{j-1} y_j}{D_j^2 + y_j^2}, \frac{x_{j+1} D_j + y_{j+1} y_j}{D_j^2 + y_j^2}, \frac{D_j y_{j+1} - x_{j+1} y_j}{D_j^2 + y_j^2}, \\ \cdots, \frac{x_n D_j + y_n y_j}{D_j^2 + y_j^2}, \frac{D_j y_n - x_n y_j}{D_j^2 + y_j^2} \Big) & \text{if } k = j \end{cases}$$

(See next pages for more.)

No.

Date

$$(\bar{X}_k^{-1}\circ f\circ \varphi_j^-)(x_0,y_0,\cdots,\hat{x}_j,y_j,\cdots,x_n,y_n) = \bar{X}_k^{-1}\left(\left[x_0+iy_0,\cdots,x_{j-1}+iy_{j-1},-D_j+iy_j,x_{j+1}+iy_{j+1},\cdots,x_n+iy_n\right]\right)$$

$$= \begin{cases} \begin{aligned} &\Big(\frac{x_0x_k+y_0y_k}{x_k^2+y_k^2},\frac{x_ky_0-x_0y_k}{x_k^2+y_k^2},\cdots,\frac{x_{k-1}x_k+y_{k-1}y_k}{x_k^2+y_k^2},\frac{x_ky_{k-1}-x_{k-1}y_k}{x_k^2+y_k^2},\frac{x_{k+1}x_k+y_{k+1}y_k}{x_k^2+y_k^2},\frac{x_ky_{k+1}-x_{k+1}y_k}{x_k^2+y_k^2},\cdots,\\ &\frac{x_{j-1}x_k+y_{j-1}y_k}{x_k^2+y_k^2},\frac{x_ky_{j-1}-x_{j-1}y_k}{x_k^2+y_k^2},\frac{-D_jx_k+y_jy_k}{x_k^2+y_k^2},\frac{x_ky_j+D_jy_k}{x_k^2+y_k^2},\frac{x_{j+1}x_k+y_{j+1}y_k}{x_k^2+y_k^2},\frac{x_ky_{j+1}-x_{j+1}y_k}{x_k^2+y_k^2},\cdots,\\ &\frac{x_nx_k+y_ny_k}{x_k^2+y_k^2},\frac{x_ky_n-x_ny_k}{x_k^2+y_k^2}\Big) \end{aligned} & \text{if } k<j \\ & (k>j \text{ similar}) \\ \begin{aligned} &\Big(\frac{-x_0D_j+y_0y_j}{D_j^2+y_j^2},-\frac{D_jy_0+x_0y_j}{D_j^2+y_j^2},\cdots,\frac{-x_{j-1}D_j+y_{j-1}y_j}{D_j^2+y_j^2},-\frac{D_jy_{j-1}+x_{j-1}y_j}{D_j^2+y_j^2},\frac{-x_{j+1}D_j+y_{j+1}y_j}{D_j^2+y_j^2},\\ &-\frac{D_jy_{j+1}+x_{j+1}y_j}{D_j^2+y_j^2},\cdots,\frac{-x_nD_j+y_ny_j}{D_j^2+y_j^2},-\frac{D_jy_n+x_ny_j}{D_j^2+y_j^2}\Big) \end{aligned} & \text{if } k=j \end{cases}$$

$$(\bar{X}_k^{-1}\circ f\circ \psi_j^+)(x_0,y_0,\cdots,x_j,\hat{y}_j,\cdots,x_n,y_n) = \bar{X}_k^{-1}\left(\left[x_0+iy_0,\cdots,x_{j-1}+iy_{j-1},x_j+iE_j,x_{j+1}+iy_{j+1},\cdots,x_n+iy_n\right]\right)$$

$$= \begin{cases} \begin{aligned} &\Big(\frac{x_0x_k+y_0y_k}{x_k^2+y_k^2},\frac{x_ky_0-x_0y_k}{x_k^2+y_k^2},\cdots,\frac{x_{k-1}x_k+y_{k-1}y_k}{x_k^2+y_k^2},\frac{x_ky_{k-1}-x_{k-1}y_k}{x_k^2+y_k^2},\frac{x_{k+1}x_k+y_{k+1}y_k}{x_k^2+y_k^2},\frac{x_ky_{k+1}-x_{k+1}y_k}{x_k^2+y_k^2},\cdots,\\ &\frac{x_{j-1}x_k+y_{j-1}y_k}{x_k^2+y_k^2},\frac{x_ky_{j-1}-x_{j-1}y_k}{x_k^2+y_k^2},\frac{x_jx_k+E_jy_k}{x_k^2+y_k^2},\frac{x_kE_j-x_jy_k}{x_k^2+y_k^2},\frac{x_{j+1}x_k+y_{j+1}y_k}{x_k^2+y_k^2},\frac{x_ky_{j+1}-x_{j+1}y_k}{x_k^2+y_k^2},\cdots,\\ &\frac{x_nx_k+y_ny_k}{x_k^2+y_k^2},\frac{x_ky_n-x_ny_k}{x_k^2+y_k^2}\Big) \end{aligned} & \text{if } k<j \\ & (k>j \text{ similar}) \\ \begin{aligned} &\Big(\frac{x_0x_j+y_0E_j}{x_j^2+E_j^2},\frac{x_jy_0-x_0E_j}{x_j^2+E_j^2},\cdots,\frac{x_{j-1}x_j+y_{j-1}E_j}{x_j^2+E_j^2},\frac{x_jy_{j-1}-x_{j-1}E_j}{x_j^2+E_j^2},\frac{x_{j+1}x_j+y_{j+1}E_j}{x_j^2+E_j^2},\\ &\frac{x_jy_{j+1}-x_{j+1}E_j}{x_j^2+E_j^2},\cdots,\frac{x_nx_j+y_nE_j}{x_j^2+E_j^2},\frac{x_jy_n-x_nE_j}{x_j^2+E_j^2}\Big) \end{aligned} & \text{if } k=j \end{cases}$$

(See next page for more.)

$$(\overline{x}_k^{-1}\circ f\circ \psi_j^-)(x_0,y_0,\cdots,x_j,\hat{y}_j,\cdots,x_n,y_n) = \overline{x}_k^{-1}\left([x_0+iy_0,\cdots,x_{j-1}+iy_{j-1},x_j+iE_j,x_{j+1}+iy_{j+1},\cdots,x_n+iy_n]\right)$$

$$= \begin{cases} \Big(\frac{x_0x_k+y_0y_k}{x_k^2+y_k^2}, \frac{x_ky_0-x_0y_k}{x_k^2+y_k^2}, \cdots, \frac{x_{k-1}x_k+y_{k-1}y_k}{x_k^2+y_k^2}, \frac{x_ky_{k-1}-x_{k-1}y_k}{x_k^2+y_k^2}, \frac{x_{k+1}x_k+y_{k+1}y_k}{x_k^2+y_k^2}, \frac{x_ky_{k+1}-x_{k+1}y_k}{x_k^2+y_k^2}, \cdots, \\ \frac{x_{j-1}x_k+y_{j-1}y_k}{x_k^2+y_k^2}, \frac{x_ky_{j-1}-x_{j-1}y_k}{x_k^2+y_k^2}, \frac{x_jx_k-E_jy_k}{x_k^2+y_k^2}, -\frac{x_kE_j+x_jy_k}{x_k^2+y_k^2}, \frac{x_{j+1}x_k+y_{j+1}y_k}{x_k^2+y_k^2}, \frac{x_ky_{j+1}-x_{j+1}y_k}{x_k^2+y_k^2}, \cdots, \\ \qquad \frac{x_nx_k+y_ny_k}{x_k^2+y_k^2}, \frac{x_ky_n-x_ny_k}{x_k^2+y_k^2}\Big) & \text{if } k<j \\ & (k>j \text{ similar}) \\ \Big(\frac{x_0x_j-y_0E_j}{x_j^2+E_j^2}, -\frac{x_jy_0+x_0E_j}{x_j^2+E_j^2}, \cdots, \frac{x_{j-1}x_j-y_{j-1}E_j}{x_j^2+E_j^2}, -\frac{x_jy_{j-1}+x_{j-1}E_j}{x_j^2+E_j^2}, \frac{x_{j+1}x_j-y_{j+1}E_j}{x_j^2+E_j^2}, \\ \qquad -\frac{x_jy_{j+1}+x_{j+1}E_j}{x_j^2+E_j^2}, \cdots, \frac{x_nx_j-y_nE_j}{x_j^2+E_j^2}, -\frac{x_jy_n+x_nE_j}{x_j^2+E_j^2}\Big) & \text{if } k=j \end{cases}$$

It is obviously checked that all the maps above are differentiable. Hence $f: S^{2n+1}\longrightarrow \mathbb{CP}^n$ as we defined above is a differentiable map.

By our construction $f(z_0,\cdots,z_n)=[z_0,\cdots,z_n]$, it follows that if $f(W_1)=f(W_2)$ then $W_1\sim W_2$ as points in $\mathbb{C}^{n+1}$. Since $W_1,W_2\in S^{2n+1}$, by our previous argument one has $W_1=e^{i\theta}W_2$ for some $\theta\in\mathbb{R}$. Hence for any two points in $f^{-1}([Z])$ there exists some $\theta\in[0,2\pi]$ relating them to each other through multiplication by $e^{i\theta}$ or $e^{-i\theta}$. Fixing an arbitrary $N\in S^{2n+1}$ such that $[N]=[Z]$ (whose existence is obviously seen if one takes $N=Z$, for example), then we have

$f^{-1}([Z])=\{e^{i\theta}N\in S^{2n+1}, N\in[Z]\cap S^{2n+1}, 0\le\theta\le 2\pi\}=[Z]\cap S^{2n+1}$

where the last equality can be obtained as follows:

$e^{i\theta}N\in S^{2n+1}$ for all $\theta\in[0,2\pi]$, and $[e^{i\theta}N]=[N]=[Z]\Rightarrow\{e^{i\theta}N\in S^{2n+1}, N\in[Z]\cap S^{2n+1}, 0\le\theta\le2\pi\}\subset[Z]\cap S^{2n+1}$

for any $W\in[Z]\cap S^{2n+1}$, by the first part of this solution to c) one has $W=e^{i\theta}N$ for some $\theta\in[0,2\pi]$ $\Rightarrow [Z]\cap S^{2n+1}\subset\{e^{i\theta}N\in S^{2n+1}, N\in[Z]\cap S^{2n+1}, 0\le\theta\le2\pi\}$. Combining both directions, one concludes that $\{e^{i\theta}N\in S^{2n+1}, N\in[Z]\cap S^{2n+1}, 0\le\theta\le2\pi\}=[Z]\cap S^{2n+1}$.

$\boxed{\text{d)}}$ We show that $f: S^{2n+1}\longrightarrow\mathbb{CP}^n$ as we defined is a submersion via direct computations of the Jacobian matrices of f in coordinates. We will only write out the Jacobian matrix for the map $\overline{x}_k^{-1}\circ f\circ\varphi_j^+ : \mathbb{R}^{2n+1}\longrightarrow\mathbb{R}^{2n}$, for all other three cases are very similar. In the remark we are going to point out another approach which involves boxed-up theorems from other materials.

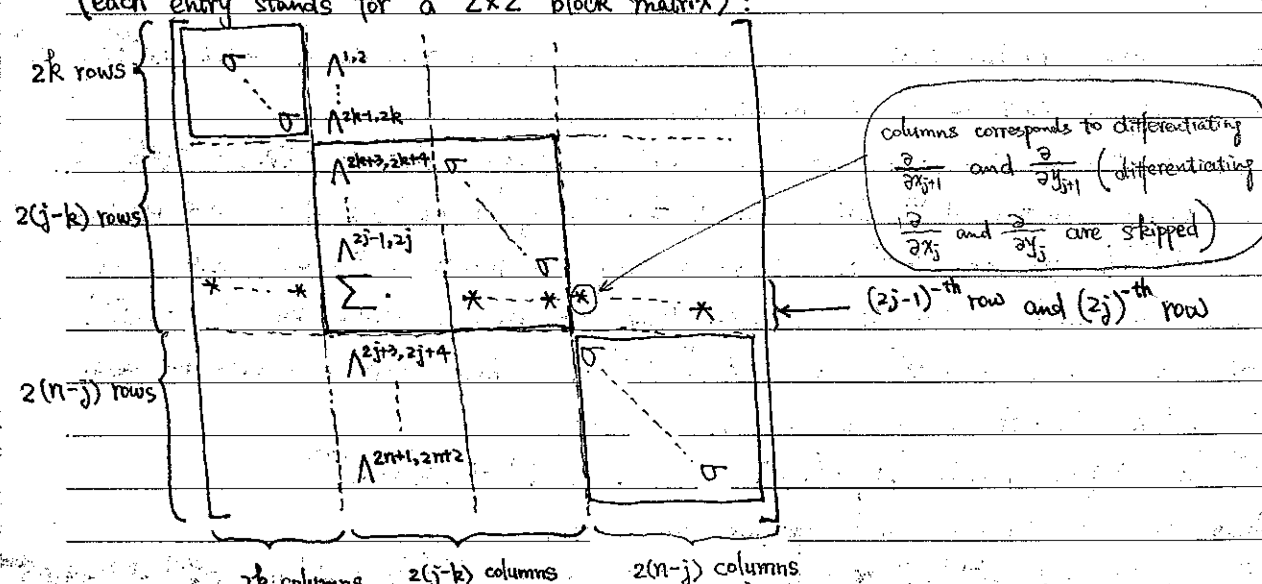
No.

Date

- First, let us look at the coordinate expression of $\bar{x}_k^{-1} \circ f \circ \varphi_j^+$ when $k<j$. The case $k>j$ is completely similar. Note that $J(\bar{x}_k^{-1} \circ f \circ \varphi_j^+)$ is a $2n \times (2n+1)$ matrix, thus in order to show $J(\bar{x}_k^{-1} \circ f \circ \varphi_j^+)$ has rank $2n$ it suffices to show the square matrix formed by $J(\bar{x}_k^{-1} \circ f \circ \varphi_j^+)$ with one row deleted has non-vanishing determinant. Observe that the entries of $\bar{x}_k^{-1} \circ f \circ \varphi_j^+$ appear in pairs, and for all $m \neq j$ we have the $\frac{\partial}{\partial x^m}$ term for each component of $\bar{x}_k^{-1} \circ f \circ \varphi_j^+$, but the $\frac{\partial}{\partial x^j}$ term is always missing. It is thus natural to throw away the $\frac{\partial}{\partial y^j}$ column as well. Denoting σ for the 2×2 block $\begin{pmatrix} \frac{x_k}{x_k^2+y_k^2} & \frac{y_k}{x_k^2+y_k^2} \\ \frac{-y_k}{x_k^2+y_k^2} & \frac{x_k}{x_k^2+y_k^2} \end{pmatrix}$ whose determinant is

$$\det \sigma = \frac{x_k^2}{(x_k^2+y_k^2)^2} + \frac{y_k^2}{(x_k^2+y_k^2)^2} = \frac{1}{x_k^2+y_k^2}.$$

Then $S := \left(J(\bar{x}_k^{-1} \circ f \circ \varphi_j^+) \text{ with } (2j+1)\text{-th column deleted} \right)$ has the following form (each entry stands for a 2×2 block matrix):



where

$$\Lambda^{2l+1,2l+2} = \begin{pmatrix} \frac{\partial}{\partial x_k}\left(\frac{x_l x_k + y_l y_k}{x_k^2+y_k^2}\right) & \frac{\partial}{\partial y_k}\left(\frac{x_l x_k + y_l y_k}{x_k^2+y_k^2}\right) \\ \frac{\partial}{\partial x_k}\left(\frac{x_k y_l - x_l y_k}{x_k^2+y_k^2}\right) & \frac{\partial}{\partial y_k}\left(\frac{x_k y_l - x_l y_k}{x_k^2+y_k^2}\right) \end{pmatrix}$$

$$= \begin{pmatrix} \frac{-x_l(x_k^2-y_k^2) - 2y_l x_k y_k}{(x_k^2+y_k^2)^2} & \frac{y_l(x_k^2-y_k^2) - 2x_l x_k y_k}{(x_k^2+y_k^2)^2} \\ \frac{-y_l(x_k^2-y_k^2) + 2x_l x_k y_k}{(x_k^2+y_k^2)^2} & \frac{-x_l(x_k^2-y_k^2) - 2y_l x_k y_k}{(x_k^2+y_k^2)^2} \end{pmatrix}$$

for $0 \leq l \leq n$, $l \neq k, j$; and

$$\Sigma = \begin{pmatrix} \frac{\partial}{\partial x_k}\left(\frac{D_j x_k + y_j y_k}{x_k^2+y_k^2}\right) & \frac{\partial}{\partial y_k}\left(\frac{D_j x_k + y_j y_k}{x_k^2+y_k^2}\right) \\ \frac{\partial}{\partial x_k}\left(\frac{x_k y_j - D_j y_k}{x_k^2+y_k^2}\right) & \frac{\partial}{\partial y_k}\left(\frac{x_k y_j - D_j y_k}{x_k^2+y_k^2}\right) \end{pmatrix}$$

$$= \begin{pmatrix} \frac{\frac{\partial D_j}{\partial x_k} x_k (x_k^2+y_k^2) - D_j(x_k^2-y_k^2) - 2y_j x_k y_k}{(x_k^2+y_k^2)^2} & \frac{\frac{\partial D_j}{\partial y_k} x_k (x_k^2+y_k^2) + y_j(x_k^2-y_k^2) - 2D_j x_k y_k}{(x_k^2+y_k^2)^2} \\ \frac{-\frac{\partial D_j}{\partial x_k} y_k (x_k^2+y_k^2) - y_j(x_k^2-y_k^2) + 2D_j x_k y_k}{(x_k^2+y_k^2)^2} & \frac{-\frac{\partial D_j}{\partial y_k} y_k (x_k^2+y_k^2) - D_j(x_k^2-y_k^2) - 2y_j x_k y_k}{(x_k^2+y_k^2)^2} \end{pmatrix}$$

where $D_j = \sqrt{1 - \sum_{\substack{l=0 \\ l\neq j}}^{n} x_l^2 - \sum_{l=0}^{n} y_l^2}$.

Before computing $\det S$, we first simplify S by clearing the $*$'s on the $(2(j-k)-1)$-th and $(2(j-k))$-th row. This is done by multiplying the $2\times(2n)$ matrix formed by the $(2l-1)$-th and $(2l)$-th row with

$$\begin{pmatrix} -\frac{\partial D_j}{\partial x_l}\frac{x_k}{x_k^2+y_k^2} & -\frac{\partial D_j}{\partial y_l}\frac{x_k}{x_k^2+y_k^2} \\ \frac{\partial D_j}{\partial x_l}\frac{y_k}{x_k^2+y_k^2} & \frac{\partial D_j}{\partial y_l}\frac{y_k}{x_k^2+y_k^2} \end{pmatrix} \begin{pmatrix} x_k & -y_k \\ y_k & x_k \end{pmatrix} \quad \text{from left}$$

and add the resulting $2\times(2n)$ matrix to the $2\times(2n)$ matrix formed by the $(2(j-k)-1)$-th and $(2(j-k))$-th row, for all $l = 0,\cdots,k-1,k+1,\cdots,j-1,j+1,\cdots,n$. After all these manipulations, all the $*$'s in S become 0 while Σ becomes

$$\tilde{\Sigma} = \Sigma + \sum_{\substack{l=0 \\ l\neq k,j}}^{n} \begin{pmatrix} -\frac{\partial D_j}{\partial x_l}\frac{x_k}{x_k^2+y_k^2} & -\frac{\partial D_j}{\partial y_l}\frac{x_k}{x_k^2+y_k^2} \\ \frac{\partial D_j}{\partial x_l}\frac{y_k}{x_k^2+y_k^2} & \frac{\partial D_j}{\partial y_l}\frac{y_k}{x_k^2+y_k^2} \end{pmatrix} \begin{pmatrix} x_k & -y_k \\ y_k & x_k \end{pmatrix} \Lambda^{2l+1,2l+2}.$$

After long and tedious computations, the final result is on the next page:

(noting that $\frac{\partial D_j}{\partial x_l} = -\frac{x_l}{D_j}$, $\frac{\partial D_j}{\partial y_l} = -\frac{y_l}{D_j}$ for $0 \le l \le n$, $l \neq k,j$

~~and that~~ $\sum_{\substack{l=0 \\ l\neq k,j}}^{n} (x_l^2+y_l^2) = 1 - D_j^2 - y_j^2 - (x_k^2+y_k^2)$)

No.

$$\tilde{\Sigma} = \Sigma + \begin{pmatrix} \dfrac{-x_k^2 + D_j^2 x_k^2 + y_j^2 x_k^2 + x_k^2(x_k^2+y_k^2)}{D_j(x_k^2+y_k^2)^2} & \dfrac{-x_k y_k + D_j^2 x_k y_k + x_k y_k y_j^2 + x_k y_k (x_k^2+y_k^2)}{D_j(x_k^2+y_k^2)^2} \\[2ex] \dfrac{x_k y_k - D_j^2 x_k y_k - x_k y_k y_j^2 - x_k y_k (x_k^2+y_k^2)}{D_j(x_k^2+y_k^2)^2} & \dfrac{y_k^2 - D_j^2 y_k^2 - y_k^2 y_j^2 - y_k^2(x_k^2+y_k^2)}{D_j(x_k^2+y_k^2)^2} \end{pmatrix}$$

$$= \frac{1}{D_j(x_k^2+y_k^2)^2} \begin{pmatrix} -x_k^2(x_k^2+y_k^2) - D_j^2(x_k^2-y_k^2) - 2D_j y_j x_k y_k & -x_k y_k(x_k^2+y_k^2) + D_j y_j (x_k^2-y_k^2) - 2D_j^2 x_k y_k \\ x_k y_k (x_k^2+y_k^2) - D_j y_j (x_k^2-y_k^2) + 2D_j^2 x_k y_k & y_k^2(x_k^2+y_k^2) - D_j^2(x_k^2-y_k^2) - 2D_j y_j x_k y_k \end{pmatrix}$$

$$+ \frac{1}{D_j(x_k^2+y_k^2)^2} \begin{pmatrix} -x_k^2 + D_j^2 x_k^2 + y_j^2 x_k^2 + x_k^2(x_k^2+y_k^2) & -x_k y_k + D_j^2 x_k y_k + x_k y_k y_j^2 + x_k y_k (x_k^2+y_k^2) \\ x_k y_k - D_j^2 x_k y_k - x_k y_k y_j^2 - x_k y_k (x_k^2+y_k^2) & y_k^2 - D_j^2 y_k^2 - y_k^2 y_j^2 - y_k^2 (x_k^2+y_k^2) \end{pmatrix}$$

$$= \frac{1}{D_j(x_k^2+y_k^2)^2} \begin{pmatrix} (D_j y_k - x_k y_j)^2 - x_k^2 & -(D_j x_k + y_j y_k)(D_j y_k - x_k y_j) - x_k y_k \\ (D_j x_k + y_j y_k)(D_j y_k - x_k y_j) + x_k y_k & -(D_j x_k + y_j y_k)^2 + y_k^2 \end{pmatrix}$$

Hence $\det \tilde{\Sigma} = \dfrac{1}{D_j^2 (x_k^2+y_k^2)^4} \det \begin{pmatrix} (D_j y_k - x_k y_j)^2 - x_k^2 & -(D_j x_k + y_j y_k)(D_j y_k - x_k y_j) - x_k y_k \\ (D_j x_k + y_j y_k)(D_j y_k - x_k y_j) + x_k y_k & -(D_j x_k + y_j y_k)^2 + y_k^2 \end{pmatrix}$

$$= \frac{1}{D_j^2(x_k^2+y_k^2)^4} \Big[\cancel{-(D_j y_k - x_k y_j)^2 (D_j x_k + y_j y_k)^2} + y_k^2 (D_j y_k - x_k y_j)^2 + x_k^2 (D_j x_k + y_j y_k)^2 \cancel{- x_k^2 y_k^2}$$
$$+ \cancel{(D_j x_k + y_j y_k)^2 (D_j y_k - x_k y_j)^2} + 2 x_k y_k (D_j x_k + y_j y_k)(D_j y_k - x_k y_j) \cancel{+ x_k^2 y_k^2} \Big]$$

$$= \frac{1}{D_j^2(x_k^2+y_k^2)^4} \Big[y_k (D_j y_k - x_k y_j) + x_k (D_j x_k + y_j y_k) \Big]^2 = \frac{1}{D_j^2 (x_k^2+y_k^2)^4} \Big[D_j (x_k^2+y_k^2) \cancel{- x_k y_k y_j} \cancel{+ x_k y_k y_j} \Big]^2$$

$$= \frac{1}{D_j^2(x_k^2+y_k^2)^4} \cdot D_j^2 (x_k^2+y_k^2)^2 = \frac{1}{(x_k^2+y_k^2)^2} > 0$$ since in the domain of $\bar{x}_k^{-1} \circ f \circ \varphi_j^+$ one has $x_k^2 + y_k^2 > 0$.

Now we are ready to compute $\det S$. Note that after the manipulations described above, all $*$'s are set to 0, with Σ replaced by $\tilde{\Sigma}$, and the manipulations here all preserve $\det S$. Further, we can perform similar manipulations to the columns, this time setting the entries taken by $\Lambda^{2l+1, 2l+2}$ $(0 \le l \le n,\ l \ne k, j)$ all to be zero; these manipulations all preserve $\det S$, and moreover leave $\tilde{\Sigma}$ unchanged since all $*$'s are already cleared in the first step. Now it follows that

$$\det S = \det \begin{pmatrix} \sigma & & \\ & \ddots & \\ & & \sigma \end{pmatrix}_{2k} \cdot \det \begin{pmatrix} 0 & \sigma & & \\ \vdots & & \ddots & \\ 0 & & & \sigma \\ \tilde{\Sigma} & 0 & \cdots & 0 \end{pmatrix} \cdot \det \begin{pmatrix} \sigma & & \\ & \ddots & \\ & & \sigma \end{pmatrix}_{2(n-j)} = (\det \sigma)^k \cdot \det \tilde{\Sigma} \, (\det \sigma)^{j-k-1} \cdot (\det \sigma)^{n-j}$$

$$= (\det \sigma)^{n-1} \cdot \det \tilde{\Sigma} = \frac{1}{(x_k^2+y_k^2)^{n-1}} \cdot \frac{1}{(x_k^2+y_k^2)^2} = \frac{1}{(x_k^2+y_k^2)^{n+1}} > 0$$ since in the domain of $\bar{x}_k^{-1} \circ f \circ \varphi_j^+$ one has $x_k^2 + y_k^2 > 0$.

Thus we conclude that $\bar{x}_k^{-1}\circ f\circ\varphi_j^+ : \mathbb{R}^{2n+1}\longrightarrow\mathbb{R}^{2n}$ is a submersion provided $k<j$. A similar argument establishes the same result for $k>j$.

- Next let us look at the coordinate expression of $\bar{x}_k^{-1}\circ f\circ\varphi_j^+$ when $k=j$. For the same reason as in the first case, we delete the $\frac{\partial}{\partial y_j}$ column in $J(\bar{x}_j^{-1}\circ f\circ\varphi_j^+)$ and denote S for the resulting $2n\times 2n$ matrix. Direct computation yields

$$S=\begin{pmatrix}\sigma_0 & & & & \\ & \ddots & & & \\ & & \sigma_{j-1} & & \\ & & & \sigma_{j+1} & \\ & & & & \ddots \\ & & & & & \sigma_n\end{pmatrix}$$

where

$$\sigma_\ell=\begin{pmatrix}\frac{\partial}{\partial x_\ell}\left(\frac{x_\ell D_j+y_\ell y_j}{D_j^2+y_j^2}\right) & \frac{\partial}{\partial y_\ell}\left(\frac{x_\ell D_j+y_\ell y_j}{D_j^2+y_j^2}\right)\\ \frac{\partial}{\partial x_\ell}\left(\frac{D_j y_\ell-x_\ell y_j}{D_j^2+y_j^2}\right) & \frac{\partial}{\partial y_\ell}\left(\frac{D_j y_\ell-x_\ell y_j}{D_j^2+y_j^2}\right)\end{pmatrix}$$

$$=\frac{1}{D_j(D_j^2+y_j^2)^2}\begin{pmatrix}(D_j^2-x_\ell^2)(D_j^2+y_j^2)+2x_\ell D_j(x_\ell D_j+y_\ell y_j) & (D_jy_j-x_\ell y_\ell)(D_j^2+y_j^2)+2y_\ell D_j(x_\ell D_j+y_\ell y_j)\\ -(D_jy_j+x_\ell y_\ell)(D_j^2+y_j^2)+2x_\ell D_j(D_jy_\ell-x_\ell y_j) & (D_j^2-y_\ell^2)(D_j^2+y_j^2)+2y_\ell D_j(D_jy_\ell-x_\ell y_j)\end{pmatrix}$$

Note that
$$\det\sigma_\ell=\frac{1}{D_j^2(D_j^2+y_j^2)^4}\Big[\big[(D_j^2-x_\ell^2)(D_j^2+y_j^2)+2x_\ell D_j(x_\ell D_j+y_\ell y_j)\big]\big[(D_j^2-y_\ell^2)(D_j^2+y_j^2)+2y_\ell D_j(D_jy_\ell-x_\ell y_j)\big]$$
$$+\big[(D_jy_j-x_\ell y_\ell)(D_j^2+y_j^2)+2y_\ell D_j(x_\ell D_j+y_\ell y_j)\big]\big[(D_jy_\ell+x_\ell y_\ell)(D_j^2+y_j^2)-2x_\ell D_j(D_jy_\ell-x_\ell y_j)\big]\Big]$$
$$=\frac{1}{D_j^2(D_j^2+y_j^2)^4}\Big[\big[(D_j^2-x_\ell^2)(D_j^2-y_\ell^2)+(D_jy_j-x_\ell y_\ell)(D_jy_j+x_\ell y_\ell)\big](D_j^2+y_j^2)^2$$
$$+2x_\ell D_j(D_j^2+y_j^2)\big[(x_\ell D_j+y_\ell y_j)(D_j^2-y_\ell^2)-(D_jy_\ell-x_\ell y_j)(D_jy_j-x_\ell y_\ell)\big]$$
$$+2y_\ell D_j(D_j^2+y_j^2)\big[(D_jy_\ell-x_\ell y_j)(D_j^2-x_\ell^2)+(x_\ell D_j+y_\ell y_j)(D_jy_j+x_\ell y_\ell)\big]\Big]$$
$$=\frac{1}{D_j^2(D_j^2+y_j^2)^4}\Big[D_j^2(D_j^2-x_\ell^2-y_\ell^2+y_j^2)(D_j^2+y_j^2)^2+2x_\ell^2D_j^2(D_j^2+y_j^2)^2-\cancel{2x_\ell y_\ell D_jy_j(x_\ell^2+y_\ell^2)(D_j^2+y_j^2)}$$
$$+2y_\ell^2D_j^2(D_j^2+y_j^2)^2+\cancel{2x_\ell y_\ell D_jy_j(x_\ell^2+y_\ell^2)(D_j^2+y_j^2)}\Big]$$
$$=\frac{1}{\cancel{D_j^2}(D_j^2+y_j^2)^4}\cdot\cancel{D_j^2}\cancel{(D_j^2+y_j^2)^2}(D_j^2+y_j^2+x_\ell^2+y_\ell^2)=\frac{D_j^2+y_j^2+x_\ell^2+y_\ell^2}{(D_j^2+y_j^2)^2}=\frac{1}{D_j^2+y_j^2}+\frac{x_\ell^2+y_\ell^2}{(D_j^2+y_j^2)^2}>0$$

where the strict inequality holds because $D_j^2+y_j^2>0$ on the domain of definition of $\bar{x}_j^{-1}\circ f\circ\varphi_j^+$.

It follows that $\det S = \prod_{\substack{l=0 \\ l\neq j}}^{n} (\det \sigma_l) = \prod_{\substack{l=0 \\ l\neq j}}^{n} \left(\frac{1}{D_j^2+y_j^2} + \frac{x_l^2+y_l^2}{(D_j^2+y_j^2)^2} \right)$

$$\geqslant \prod_{\substack{l=0 \\ l\neq j}}^{n} \frac{1}{D_j^2+y_j^2} = \frac{1}{(D_j^2+y_j^2)^n} > 0$$

since $D_j^2+y_j^2>0$ on the domain of definition of $\bar{x}_j^{-1}\circ f\circ \varphi_j^+$. Thus we conclude that $\bar{x}_j^{-1}\circ f\circ \varphi_j^+ : \mathbb{R}^{2n+1} \longrightarrow \mathbb{R}^{2n}$ is a submersion.

- Repeating the argument above for all $\bar{x}_k^{-1}\circ f\circ \varphi_j^-$, $\bar{x}_k^{-1}\circ f\circ \psi_j^+$, $\bar{x}_k^{-1}\circ f\circ \psi_j^-$ for all $0\leq k, j\leq n$. Hence the differentiable map $f: S^{2n+1}\longrightarrow \mathbb{CP}^n$ we gave in c) is a submersion.

Remark: There is an alternative way to do d). The following theorem may help (cf. John Lee "Introduction to Smooth Manifolds" P153 Theorem 7.10):

Quotient Manifold Theorem. Suppose a Lie group G acts smoothly, freely, and properly on a smooth manifold M. Then the orbit space M/G is a topological manifold of dimension equal to $\dim M - \dim G$, and has a unique smooth structure with the property that the quotient map $\pi: M\longrightarrow M/G$ is a smooth submersion.

If there is only one differentiable structure on $\mathbb{CP}^n$, we are done; but this is not likely true (think of various non-equivalent differentiable structures on spheres of certain dimensions). Thus one may read the proof to the theorem above and check to see if the differentiable structure specified in the theorem (which comes from rank theorems essentially) is compatible with the differentiable structure we constructed in a). For another reference, see S. Gallot, D. Hulin, J. Lafontaine "Riemannian Geometry" P32 Theorem 1.95 and Exercise 1.96, which focus on $\mathbb{CP}^n$ being diffeomorphic to S^{2n+1}/S^1, and point readers to W. Boothby "An Introduction to Differentiable Manifolds and Riemannian Geometry".

12. (Curvature of the complex projective space). Define a Riemannian metric on $\mathbb{C}^{n+1}\backslash\{0\}$ in the following way. If $Z \in \mathbb{C}^{n+1}\backslash\{0\}$ and $V, W \in T_Z(\mathbb{C}^{n+1}\backslash\{0\})$,

$$\langle V, W\rangle_Z = \frac{\mathrm{Real}(V,W)}{(Z,Z)}.$$

Observe that the metric $\langle\,,\rangle$ restricted to $S^{2n+1} \subset \mathbb{C}^{n+1}\backslash\{0\}$ coincides with the metric induced from $\mathbb{R}^{2n+2}$.

a) Show that, for all $0 \le \theta \le 2\pi$, $e^{i\theta}: S^{2n+1} \longrightarrow S^{2n+1}$ is an isometry, and that, therefore, it is possible to define a Riemannian metric on $\mathbb{C}P^n$ in such a way that the submersion f is Riemannian.

b) Show that, in this metric, the sectional curvature of $\mathbb{C}P^n$ is given by

$$K(\sigma) = 1 + 3\cos^2\varphi,$$

where σ is generated by the orthonormal pair X, Y, $\cos\varphi = \langle \bar{X}, i\bar{Y}\rangle$, and $\bar{X}, \bar{Y}$ are the horizontal lifts of X and Y, respectively. In particular, $1 \le K(\sigma) \le 4$.

Hint for (b): Let Z be the position vector describing S^{2n+1}. Since $\left(\frac{d}{d\theta} e^{i\theta}Z\right)_{\theta=0} = iZ$, $iZ \in T_Z(S^{2n+1})$ and is vertical. Let $\bar{\nabla}$ be the Riemannian connection of $\mathbb{R}^{2n+2} \approx \mathbb{C}^{n+1}$ and $X, Y \in \mathfrak{X}(\mathbb{C}P^n)$. Take $\alpha: (-\varepsilon,\varepsilon) \longrightarrow S^{2n+1}$ with $\alpha(0) = Z$, $\alpha'(0) = \bar{X}$. Then $(\bar{\nabla}_{\bar{X}} iZ)_Z = \frac{d}{dt}(iZ\circ\alpha(t))\big|_{t=0} = \frac{d}{dt} i\alpha(t) = i\alpha'(0) = i\bar{X}$. Therefore, $\langle [\bar{X},\bar{Y}], iZ\rangle = \langle \bar{\nabla}_{\bar{X}}\bar{Y} - \bar{\nabla}_{\bar{Y}}\bar{X}, iZ\rangle = -\langle i\bar{X},\bar{Y}\rangle + \langle i\bar{Y},\bar{X}\rangle = 2\cos\varphi$. Now use Exercise 10 (b).

• $\boxed{a)}$ Let $p \in S^{2n+1}$ be an arbitrary point, and denote θ for the action of $e^{i\theta}$ on S^{2n+1}, i.e. $\theta(p) = e^{i\theta}p$. For any $V, W \in T_pS^{2n+1}$, one has $e^{i\theta}$ obviously a diffeomorphism on S^{2n+1}, and $\langle d\theta_p(V), d\theta_p(W)\rangle_{\theta(p)} = \langle e^{i\theta}V, e^{i\theta}W\rangle_{\theta(p)} = \dfrac{\mathrm{Real}(e^{i\theta}V, e^{i\theta}W)}{(\theta(p),\theta(p))} = \mathrm{Real}(e^{i\theta}e^{-i\theta}(V,W))$

$$= \mathrm{Real}(V,W)$$

$$= \frac{\mathrm{Real}(V,W)}{(p,p)} = \langle V,W\rangle_p$$

thus $e^{i\theta}: S^{2n+1} \longrightarrow S^{2n+1}$ is an isometry for all $0 \le \theta \le 2\pi$.

Note that Exercise 11 c) in this Chapter tells us that $\mathbb{C}P^n$ can be viewed as S^{2n+1}/S^1, where S^1 acts on S^{2n+1} by isometries $e^{i\theta}$ as above. $f: S^{2n+1} \longrightarrow \mathbb{C}P^n = S^{2n+1}/S^1$ is then simply sending a point $w \in S^{2n+1}$ to its orbit under the action of S^1. The differentiable structure we put on $\mathbb{C}P^n = S^{2n+1}/S^1$ then makes the canonical projection $f: S^{2n+1} \longrightarrow S^{2n+1}/S^1$ a submersion. For any $q \in \mathbb{C}P^n = S^{2n+1}/S^1$, let $\bar{q} \in S^{2n+1}$ be an arbitrary point such that $f(\bar{q}) = q$. For any $V, W \in T_q(\mathbb{C}P^n)$, let $\bar{V}, \bar{W} \in T_{\bar{q}}S^{2n+1}$ be the canonical horizontal lift of V, W respectively (note that the definition of a horizontal lift does not need the base manifold (in our case $\mathbb{CP}^n$) to possess a Riemannian metric in advance; it suffices if the domain manifold (in our case S^{2n+1}) has a Riemannian metric, and the differentiable map is a submersion). Define a metric on $\mathbb{CP}^n$ by

$$\langle V, W\rangle_q = \langle \bar{V}, \bar{W}\rangle_{\bar{q}}, \quad \bar{q} \in f^{-1}(q).$$

To see this is well-defined, we need to show that $\langle\,,\rangle_q$ is independent of the choice of $\bar{q} \in f^{-1}(q)$. Let $\bar{q}_1, \bar{q}_2$ be arbitrary points in $f^{-1}(q) \subset S^{2n+1}$. By definition of f, there exists some $\theta \in [0, 2\pi]$ such that $\bar{q}_1 = e^{i\theta}\bar{q}_2$. Since we have shown that $e^{i\theta}: S^{2n+1} \longrightarrow S^{2n+1}$ is an isometry, one has $\langle \bar{V}_1, \bar{W}_1\rangle_{\bar{q}_1} = \langle \bar{V}_2, \bar{W}_2\rangle_{\bar{q}_2}$ where $\bar{V}_1, \bar{V}_2$ are canonical horizontal lifts of V to $T_{\bar{q}_1}S^{2n+1}$ and $T_{\bar{q}_2}S^{2n+1}$ respectively, and $\bar{W}_1, \bar{W}_2$ are canonical horizontal lifts of W to $T_{\bar{q}_1}S^{2n+1}$ and $T_{\bar{q}_2}S^{2n+1}$ respectively. Thus $\langle\,,\rangle_q$ is well-defined, and obviously is symmetric, bilinear and positively definite. That it varies smoothly on $\mathbb{CP}^n$ follows from Exercise 9 a) in this Chapter: given $X, Y \in \mathfrak{X}(\mathbb{CP}^n)$, by Exercise 9 a) of Chapter 8, $\bar{X}$ and $\bar{Y}$ are smooth vectors on S^{2n+1}, and the smoothness of $\langle X, Y\rangle$ follows from the smoothness of $\langle \bar{X}, \bar{Y}\rangle^{S^{2n+1}}$ as a function on S^{2n+1}. Finally, this metric as we constructed on $\mathbb{CP}^n$ makes $f: S^{2n+1} \longrightarrow \mathbb{CP}^n$ a Riemannian submersion, since for any $p \in S^{2n+1}$ and $V, W \in T_pS^{2n+1}$ horizontal one has

$$\langle V, W\rangle_p^{S^{2n+1}} = \left(df_p(V), df_p(W)\right)_{f(p)}^{\mathbb{CP}^n}$$ and thus df_p preserves lengths of horizontal vectors if we let $W = V$.

$\boxed{b)}$ Since $\dim S^{2n+1} - \dim \mathbb{CP}^n = 2n+1-2n = 1$, the fiber $f^{-1}([z])$ over each $[z] \in \mathbb{CP}^n$ is 1-dimensional. Let $z \in f^{-1}([z]) \subset S^{2n+1}$, we want to find a non-trivial vertical tangent vector at $z \in S^{2n+1}$, and this will determine all the horizontal tangent vectors as well since $\dim(T_z S^{2n+1})^v = 1$. By Exercise 11 c) in this Chapter, we have

$$f^{-1}([z]) = \{e^{i\theta}z \in S^{2n+1},\ 0 \le \theta \le 2\pi\}$$

and hence $\gamma: (-\varepsilon, \varepsilon) \ni t \longmapsto e^{it}z$ is a smooth curve in $f^{-1}([z])$ passing through z at $t=0$. Taking the derivative at $t=0$, one has $\gamma'(0) = \frac{d}{dt}\big|_{t=0}(e^{it}z) = iz$. Note that the metric on S^{2n+1} is induced from $\mathbb{R}^{2n+2}$, hence $\langle iz, iz\rangle = z\bar{z} = 1$, i.e. iz is nontrivial as a tangent vector at $z \in S^{2n+1}$. Thus we find iz as a basis for the vertical tangent subspace $(T_z S^{2n+1})^v$.

Now let $\bar{\nabla}$ be the Riemannian connection of $\mathbb{R}^{2n+2} \approx \mathbb{C}^{n+1}$ and $X, Y \in \mathfrak{X}(\mathbb{CP}^n)$. Denote $\bar{X}, \bar{Y}$ for the horizontal lifts of X, Y respectively. Let $\alpha: (-\varepsilon, \varepsilon) \longrightarrow S^{2n+1}$ be a smooth curve on S^{2n+1} such that $\alpha(0) = z$ and $\alpha'(0) = \bar{X}|_{T_z S^{2n+1}}$. Note that $(\mathbb{R}^{2n+2}, \bar{\nabla})$ is flat, thus $(\bar{\nabla}_{\bar{X}}(iz))|_{T_z S^{2n+1}} = \bar{X}(iz)|_{T_z S^{2n+1}} = \frac{d}{dt}\Big|_{t=0} i\alpha(t) = i\alpha'(0) = i\bar{X}|_{T_z S^{2n+1}}$.

Therefore

$$\langle [\bar{X},\bar{Y}], iz\rangle = \langle \bar{\nabla}_{\bar{X}}\bar{Y} - \bar{\nabla}_{\bar{Y}}\bar{X}, iz\rangle = \langle \bar{\nabla}_{\bar{X}}\bar{Y}, iz\rangle - \langle \bar{\nabla}_{\bar{Y}}\bar{X}, iz\rangle =$$
$$= \bar{X}\langle \bar{Y}, iz\rangle - \langle \bar{Y}, \bar{\nabla}_{\bar{X}}(iz)\rangle - \bar{Y}\langle \bar{X}, iz\rangle + \langle \bar{X}, \bar{\nabla}_{\bar{Y}}(iz)\rangle$$
$$= -\langle \bar{Y}, \bar{\nabla}_{\bar{X}}(iz)\rangle + \langle \bar{X}, \bar{\nabla}_{\bar{Y}}(iz)\rangle = -\langle \bar{Y}, i\bar{X}\rangle + \langle \bar{X}, i\bar{Y}\rangle = -\langle -i\bar{Y}, \bar{X}\rangle + \langle \bar{X}, i\bar{Y}\rangle$$
$$= \langle i\bar{Y}, \bar{X}\rangle + \langle \bar{X}, i\bar{Y}\rangle = \cos\varphi + \cos\varphi = 2\cos\varphi,$$

and by our previous argument, with the fact that $|iz| = 1$, we have

$$[\bar{X},\bar{Y}]^v = \langle [\bar{X},\bar{Y}]^v, iz\rangle \cdot iz = \langle [\bar{X},\bar{Y}], iz\rangle \cdot iz = (2\cos\varphi)\cdot iz.$$

By Exercise 10 b) in this chapter, we conclude that

$$K(\sigma) = \bar{K}(\bar{\sigma}) + \tfrac{3}{4}\langle [\bar{X},\bar{Y}]^v, [\bar{X},\bar{Y}]^v\rangle = 1 + \tfrac{3}{4}\cdot(2\cos\varphi)^2\langle iz, iz\rangle = 1 + 3\cos^2\varphi.$$

In particular, $1 \leq K(\sigma) \leq 4$ since $0 \leq \cos^2\varphi \leq 1$.

Remark 1: By writing $\langle \bar{X}, i\bar{Y}\rangle = \cos\varphi$ we used the Cauchy-Schwarz Inequality that $|\langle \bar{X}, i\bar{Y}\rangle| \leq |\bar{X}|\cdot|i\bar{Y}| = 1\cdot 1 = 1$, where we used the orthonormality of X, Y and the Riemannian Submersion $|\bar{X}| = |X| = 1$, $|\bar{Y}| = |Y| = 1$ when $\bar{X}, \bar{Y}$ horizontal (note that the orthogonality is preserved by the Riemannian submersion, i.e. $\langle \bar{X}, \bar{Y}\rangle = \langle X, Y\rangle = 0$; but $\langle \bar{X}, i\bar{Y}\rangle$ is different; because $i\bar{Y}$ need not be horizontal at all (the inner product is the one on $\mathbb{R}^{2n+2}$)).

Remark 2: In b) we used the seemingly awkward notation $\bar{X}|_{T_z S^{2n+1}}$. We did this for a reason: when X is viewed as a smooth vector field on S^{2n+1}, z is read as the position vector describing S^{2n+1}, and we are supposed to write $\bar{X}(z)$ for $\bar{X}|_{T_z S^{2n+1}}$; however, when we write $\bar{\nabla}_{\bar{X}} z$, z is viewed as a smooth vector field on $\mathbb{R}^{2n+2}$, or more precisely, a smooth vector field on $\mathbb{R}^{2n+2}$ restricted to S^{2n+1}, thus z is a normal vector field on S^{2n+1}. In the meanwhile, $\bar{X}$ is viewed as a smooth vector field on $\mathbb{R}^{2n+2}$ restricted to S^{2n+1}, and $\bar{\nabla}$ takes place in $\mathbb{R}^{2n+2}$. In this setting, since $(\mathbb{R}^n, \bar{\nabla})$ is flat, $\bar{\nabla}_{\bar{X}} z = \bar{X}(z)$, and one sees a fatal confusion of notations for "$\bar{X}(z)$", especially when we are going to use $\bar{X}(iz) = i\bar{X}(z)$, which is extremely dangerous. So we have to turn to awkward notations so as to avoid such confusion: at least $\bar{X}|_{T_z S^{2n+1}}$ makes sense as it denotes the value of the section $\bar{X} \in \mathfrak{X}(S^{2n+1}) = \Gamma(S^{2n+1}; TS^{2n+1})$ on the fiber $T_z S^{2n+1}$.

No.

Date

13. Let $p \in M$ and let $\sigma : M \to M$ be an isometry such that $\sigma(p) = p$ and $d\sigma_p(v) = -v$, for all $v \in T_pM$. Let X be a parallel field along a geodesic γ in M with $\gamma(0) = p$. Show that $d\sigma_{\gamma(t)} X(\gamma(t)) = -X(\gamma(-t))$.

Hint: It is clear that $\sigma(\gamma(t)) = \gamma(-t)$. Prove that $d\sigma_{\gamma(t)} X(\gamma(t))$ is a parallel field along $\gamma(-t)$. Note that for $t=0$, $d\sigma_{\gamma(t)} X(\gamma(t)) = -X(\gamma(-t))$, and use the uniqueness of parallel fields, with given initial data.

Proof: Since $\sigma(\gamma(0)) = \sigma(p) = p$ and $\frac{d}{dt}\Big|_{t=0} \sigma(\gamma(t)) = d\sigma_{\gamma(0)}\gamma'(0) = d\sigma_p(\gamma'(0)) = -\gamma'(0)$, $t \mapsto \sigma(\gamma(t))$ is a smooth curve on M passing through p at $t=0$ with initial tangent vector $-\gamma'(0) \in T_pM$. Since γ is a geodesic and σ is an isometry, $t \mapsto \sigma(\gamma(t))$ is a geodesic on M passing through p at $t=0$ with $\frac{d}{dt}\Big|_{t=0}\sigma(\gamma(t)) = -\gamma'(0)$. By the uniqueness of geodesics passing through p with initial tangent vector, we have locally $\sigma(\gamma(t)) = \gamma(-t)$ in a neighborhood around p in M. By tracing the maximal interval of t on which $\sigma(\gamma(t)) = \gamma(-t)$, together with a connectivity argument, we can conclude that $\sigma(\gamma(t)) = \gamma(-t)$ holds wherever both sides of the equality are defined.

Now that X is a parallel field along the geodesic γ, $(\nabla_{\gamma'}X)(\gamma(t)) \equiv 0$. By the Koszul Formula, for any $Y \in T_{\gamma(t)}M$ one has

$$0 = \langle \nabla_{\gamma'}X, Y\rangle_{\gamma(t)} = \frac{1}{2}\Big\{X\langle\gamma', Y\rangle_{\gamma(t)} + \gamma'\langle Y, X\rangle_{\gamma(t)} - Y\langle X, \gamma'\rangle_{\gamma(t)} - \langle[X,Y],\gamma'\rangle_{\gamma(t)} - \langle[\gamma',Y],X\rangle_{\gamma(t)} - \langle[X,\gamma'],Y\rangle_{\gamma(t)}\Big\}$$

$$= \frac{1}{2}\Big\{(d\sigma_{\gamma(t)}X)\langle d\sigma_{\gamma(t)}(\gamma'), d\sigma_{\gamma(t)}(Y)\rangle_{\sigma(\gamma(t))} + (d\sigma_{\gamma(t)}\gamma')\langle d\sigma_{\gamma(t)}(Y), d\sigma_{\gamma(t)}(X)\rangle_{\sigma(\gamma(t))}$$

$$- (d\sigma_{\gamma(t)}Y)\langle d\sigma_{\gamma(t)}(X), d\sigma_{\gamma(t)}(\gamma')\rangle_{\sigma(\gamma(t))} - \langle[d\sigma_{\gamma(t)}(X), d\sigma_{\gamma(t)}(Y)], d\sigma_{\gamma(t)}(\gamma')\rangle_{\sigma(\gamma(t))}$$

$$- \langle[d\sigma_{\gamma(t)}(\gamma'), d\sigma_{\gamma(t)}(Y)], d\sigma_{\gamma(t)}(X)\rangle_{\sigma(\gamma(t))} - \langle[d\sigma_{\gamma(t)}(X), d\sigma_{\gamma(t)}(\gamma')], d\sigma_{\gamma(t)}(Y)\rangle_{\sigma(\gamma(t))}\Big\}$$

$$= \langle \nabla_{d\sigma_{\gamma(t)}(\gamma')} d\sigma_{\gamma(t)}(X), d\sigma_{\gamma(t)}(Y)\rangle_{\sigma(\gamma(t))} = \langle \nabla_{-\gamma'} d\sigma_{\gamma(t)}(X), d\sigma_{\gamma(t)}(Y)\rangle_{\sigma(\gamma(t))}$$

$$= \langle -\nabla_{\gamma'} d\sigma_{\gamma(t)}(X), d\sigma_{\gamma(t)}(Y)\rangle_{\sigma(\gamma(t))} = -\langle \nabla_{\gamma'}(d\sigma_{\gamma(t)}(X)), d\sigma_{\gamma(t)}(Y)\rangle_{\sigma(\gamma(t))}$$

where we used $(d\sigma_{\gamma(t)}(X))(\sigma(\gamma(t)))\big[\langle d\sigma_{\gamma(t)}(\gamma'), d\sigma_{\gamma(t)}(Y)\rangle_{\sigma(\gamma(t))}\big] = (d\sigma_{\gamma(t)}(X))(\sigma(\gamma(t)))\big[\langle\gamma', Y\rangle_{\gamma(t)}\big] =$
$= X\big[(\langle\gamma', Y\rangle_{\gamma(t)} \circ \sigma)(\gamma(t))\big] = X\big[\langle d\sigma_{\gamma(t)}(\gamma'), d\sigma_{\gamma(t)}(Y)\rangle_{\sigma(\gamma(t))}\big] = X\langle\gamma', Y\rangle_{\gamma(t)}$, etc.; and $d\sigma_{\gamma(t)}([X,\gamma']) = [d\sigma_{\gamma(t)}(X), d\sigma_{\gamma(t)}(\gamma')]$, etc.; and $\sigma(\gamma(t)) = \gamma(-t) \Rightarrow d\sigma_{\gamma(t)}(\gamma'(t)) = -\gamma'(t)$.

Since $\sigma : M \to M$ is an isometry, $d\sigma_{\gamma(t)}(Y)$ varies through all elements in $T_{\sigma(\gamma(t))}M$ for fixed $t \in \mathbb{R}$ as Y varies through all elements in $T_{\gamma(t)}M$. Thus the long equalities above yields $\langle \nabla_{\gamma'}(d\sigma_{\gamma(t)}(X)), W\rangle_{\sigma(\gamma(t))} = 0$ for all $W \in T_{\sigma(\gamma(t))}M$, i.e. $\nabla_{\gamma'}(d\sigma_{\gamma(t)}(X)) \equiv 0$. Hence $d\sigma_{\gamma(t)}X(\gamma(t))$ is a parallel field along $t \mapsto \sigma(\gamma(t)) = \gamma(-t)$. Note that $(d\sigma_{\gamma(t)}X(\gamma(t)))\big|_{t=0} = d\sigma_{\gamma(0)}X(\gamma(0)) =$
$= d\sigma_p(X(\gamma(0))) = -X(\gamma(0)) = -X(\gamma(-0))$, and that $-X(\gamma(-t))$ is also a parallel field along $t \mapsto \gamma(-t)$, we conclude from the uniqueness theorem of parallel transporting a given tangent vector along a given smooth curve (cf. Proposition 2.6 of Chapter 2, P52) that $d\sigma_{\gamma(t)}X(\gamma(t)) = -X(\gamma(-t))$ whenever both sides of the equality makes sense. ∎

14. (Geometric characterization of locally symmetric spaces). Let M be a Riemannian manifold. A local symmetry at $p \in M$ is a map $\sigma : B_\varepsilon(p) \longrightarrow B_\varepsilon(p)$ of a normal geodesic ball centered at p such that $\sigma(\gamma(t)) = \gamma(-t)$, where γ is a radial geodesic ($\gamma(0) = p$) of $B_\varepsilon(p)$. Prove that: M is locally symmetric $\Leftrightarrow$ every local symmetry is an isometry.

Hint $\Rightarrow$: Consider a geodesic frame $e_1, \cdots, e_n$ in the ball $B_\varepsilon(p)$ (Cf. Exercise 7 of Chap. 3) and put $R_{ijk\ell} = R(e_i, e_j, e_k, e_\ell)$. Since $\nabla R = 0$, $R_{ijk\ell}$ is constant along the geodesics which start from p. Let $i : T_pM \longrightarrow T_pM$ be a linear isometry given by $i(v) = -v$, $v \in T_pM$, and observe that $\sigma = \exp_p \circ i \circ \exp_p^{-1}$. Use Cartan's Theorem to establish that σ is an isometry.

Hint $\Leftarrow$: Let $p \in M$ and $Z \in T_pM$. Consider a geodesic $\gamma : (-\varepsilon, \varepsilon) \longrightarrow M$ with $\gamma(0) = p$, $\gamma'(0) = Z$. Take an orthonormal basis $e_1, \cdots, e_n$ in T_pM, and obtain, by parallel transport, a frame $e_1(t), \cdots, e_n(t)$ along γ. Put $R_{ijk\ell}(t) = R(e_i(t), e_j(t), e_k(t), e_\ell(t))$. Then

$$(\nabla_Z R)(p) = \frac{d}{dt}\Big|_{t=0} R_{ijk\ell} = \lim_{t\to 0} \frac{R_{ijk\ell}(t) - R_{ijk\ell}(-t)}{2t} = 0.$$

where, in the last equality, we use that σ is an isometry and we also use the last exercise. Since p and Z are arbitrary, $\nabla R = 0$.

- Proof: "$\Rightarrow$": Let $e_1, \cdots, e_n$ be a geodesic frame in $B_\varepsilon(p)$, a geodesic ball centering at p with radius $\varepsilon > 0$. Let $\gamma : (-\delta, \delta) \longrightarrow B_\varepsilon(p)$ be a geodesic in $B_\varepsilon(p)$ with $\gamma(0) = p$. Since M is locally symmetric, $\nabla R = 0$ (cf. Exercise 6 of Chapter 4, $P_{105-106}$), and one has for any e_i, e_j, e_k, e_ℓ

$$0 = \nabla R(e_i, e_j, e_k, e_\ell, \gamma') = \gamma'(R(e_i, e_j, e_k, e_\ell)) - R(\nabla_{\gamma'} e_i, e_j, e_k, e_\ell) - R(e_i, \nabla_{\gamma'} e_j, e_k, e_\ell)$$
$$- R(e_i, e_j, \nabla_{\gamma'} e_k, e_\ell) - R(e_i, e_j, e_k, \nabla_{\gamma'} e_\ell) = \gamma'(R(e_i, e_j, e_k, e_\ell)) = \gamma'(R_{ijk\ell}(t))$$

i.e. $R_{ijk\ell}$ is constant along the geodesics which start from p. Let $i : T_pM \longrightarrow T_pM$ be a linear isometry given by $i(v) = -v$, $v \in T_pM$. By the local uniqueness of geodesics in $B_\varepsilon(p)$ emanating from p, we observe that $\sigma = \exp_p \circ i \circ \exp_p^{-1}$ by definition, where σ is a local symmetry on $B_\varepsilon(p)$ as defined in the problem. Recall from the Solution to Exercise 7 of Chapter 3, which tells us that $e_j(t) = (d\exp_p)_{t\gamma'(0)}(e_j(0))$, where $e_j(t) = e_j \circ \gamma(t)$. Thus

$$d\sigma_{\gamma(t)}(e_j(t)) = \left((d\exp_p)_{i(t\gamma'(0))} \circ di_{t\gamma'(0)} \circ \left[(d\exp_p)_{t\gamma'(0)} \right]^{-1} \right)(e_j(t)) =$$
$$= \left((d\exp_p)_{i(t\gamma'(0))} \circ di_{t\gamma'(0)} \right)(e_j(0)) = (d\exp_p)_{-t\gamma'(0)}(-e_j(0)) = -e_j(-t) \quad \text{for all } 1 \le j \le n.$$

Now that $R_{ijk\ell}$ is constant along the geodesic γ, we have

$$\langle R(e_i(t), e_j(t)) e_k(t), e_\ell(t) \rangle_{\gamma(t)} = R_{ijk\ell}(t) = R_{ijk\ell}(-t) = \langle R(e_i(-t), e_j(-t)) e_k(-t), e_\ell(-t) \rangle_{\gamma(-t)} =$$
$$= \langle R(-e_i(-t), -e_j(-t))(-e_k(-t)), -e_\ell(-t) \rangle_{\gamma(-t)} = \langle R(d\sigma_{\gamma(t)}(e_i(t)), d\sigma_{\gamma(t)}(e_j(t)))(d\sigma_{\gamma(t)}(e_k(t))), d\sigma_{\gamma(t)}(e_\ell(t)) \rangle_{\sigma(\gamma(t))},$$

No.

Date

By Cartan's Theorem (Theorem 2.1 of Chapter 8, P157) and the explanatory words right after it, we know $\sigma: B_\varepsilon(p) \longrightarrow B_\varepsilon(p)$ is an isometry.

"$\Leftarrow$": Let $p \in M$ and $Z \in T_pM$. Consider a geodesic $\gamma: (-\varepsilon, \varepsilon) \longrightarrow M$ with $\gamma(0) = p$, $\gamma'(0) = Z$. Take an orthonormal basis $e_1, \cdots, e_n$ in T_pM, and obtain, by parallel transport, a frame $e_1(t), \cdots, e_n(t)$ along γ. Put $R_{ijkl}(t) = R(e_i(t), e_j(t), e_k(t), e_l(t))$. By our assumption, any local symmetry $\sigma: B_\varepsilon(p) \longrightarrow B_\varepsilon(p)$ is an isometry. By Exercise 13 in this Chapter, $d\sigma_{\gamma(t)} X(\gamma(t)) = -X(\gamma(-t))$ for any parallel vector field along the geodesic γ, hence

$$R_{ijkl}(t) = R(e_i(t), e_j(t), e_k(t), e_l(t)) = R(d\sigma_{\gamma(t)}(e_i(t)), d\sigma_{\gamma(t)}(e_j(t)), d\sigma_{\gamma(t)}(e_k(t)), d\sigma_{\gamma(t)}(e_l(t)))$$

$$= R(-e_i(-t), -e_j(-t), -e_k(-t), -e_l(-t)) = R(e_i(-t), e_j(-t), e_k(-t), e_l(-t)) = R_{ijkl}(-t)$$

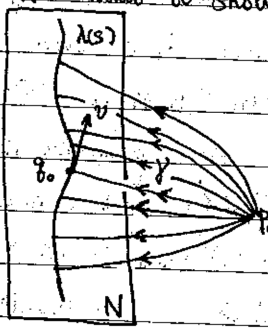
and it follows that $(\nabla_Z R)(p) = \frac{d}{dt}\Big|_{t=0} R_{ijkl}(t) = \lim_{t \to 0} \frac{R_{ijkl}(t) - R_{ijkl}(-t)}{2t} = 0$. By the arbitrary choices of $p \in M$ and $Z \in T_pM$,

this implies $\nabla R \equiv 0$ on M.

1. Let M be a complete Riemannian manifold, and let $N \subset M$ be a closed submanifold of M. Let $p_0 \in M$, $p_0 \notin N$, and let $d(p_0, N)$ be the distance from p_0 to N. Show that there exists a point $q_0 \in N$ such that $d(p_0, q_0) = d(p_0, N)$ and that a minimizing geodesic which joins p_0 to q_0 is orthogonal to N at q_0.

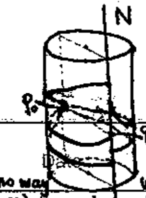
- Proof: $\boxed{1^\circ}$ By the Hopf-Rinow Theorem, (M, d) is a complete metric space. Since N is a closed subspace of M (here we use Proposition 2.6 in Chapter 7 to see that a closed submanifold is also a closed subspace of M), and that $p_0 \notin N$, we know from standard functional analysis that $d(p_0, N) > 0$. Recall that $d(p_0, N) = \inf_{q \in N} d(p_0, q)$, we can find a sequence $\{q_i\}_{i=1}^{\infty}$ of points in the manifold M such that $\{d(p_0, q_i)\}_{i=1}^{\infty}$ is a Cauchy sequence of real numbers. Note that fixing an arbitrary $\varepsilon_0 > 0$ we can find an integer $N \in \mathbb{N}$ such that $|d(p_0, q_m) - d(p_0, N)| < \varepsilon_0$ for all $n > N$, which implies that the sequence $\{q_i\}_{i=1}^{\infty}$ can be contained in a closed ball $B_r(p_0) \subset M$ with finite radius $r = \max\{d(p_0, q_1), \cdots, d(p_0, q_N), \varepsilon_0 + d(p_0, N)\}$. By the Hopf-Rinow Theorem, $B_r(p_0)$ is compact, and thus $\{q_i\}_{i=1}^{\infty}$ has a convergent subsequence converging to some $q_0 \in \overline{B_r(p_0)} = B_r(p_0) \subset M$. By passing to a subsequence if necessary, we may still write $q_i \longrightarrow q_0$ as $i \longrightarrow \infty$ for $q_0 \in M$. Since N is closed, $q_0 \in N$, and it follows from the triangle inequality that $|d(p_0, q_0) - d(p_0, N)| \leq |d(p_0, q_0) - d(p_0, q_i)| + |d(p_0, N) - d(p_0, q_i)| \longrightarrow 0$ as $i \longrightarrow \infty$, i.e. $d(p_0, q_0) = d(p_0, N)$.

$\boxed{2^\circ}$ For simplicity of notations, let $l := d(p_0, q_0)$. Let $\gamma : [0, l] \longrightarrow M$ be a normalized minimizing geodesic joining p_0 to q_0, whose existence is ensured by the completeness of M. We want to show that $\langle \gamma'(l), v \rangle_{q_0}^{M} = 0$ for all $v \in T_{q_0}N$. For this purpose, we first



construct a curve $\lambda : (-\varepsilon, \varepsilon) \longrightarrow N \subset M$ satisfying $\lambda(0) = q_0$, $\lambda'(0) = v$. The reason that we can make this curve completely lying in N is that $v \in T_{q_0}N \subset T_{q_0}M$. There are two cases to be taken into considerations here: either p_0 and q_0 are conjugate along the minimizing geodesic $\gamma : [0, l] \longrightarrow M$, or they are not. Let us first consider the case in which p_0, q_0 are not conjugate along γ. In this case, $\exp_{p_0} : T_{p_0}M \longrightarrow M$ is a local diffeomorphism around $l\gamma'(0) \in T_{p_0}M$, by Proposition 3.5 in Chapter 5 (P.117). Hence there exists a neighborhood V of $l\gamma'(0) \in T_{p_0}M$ and a neighborhood W of $q_0 \in M$ such that $\exp_{p_0}|_V : V \longrightarrow M$ is a diffeomorphism. In particular, for any $\lambda(s) \in \lambda \cap W$, there exists $W(s) \in V \subset T_{p_0}M$ such that $\lambda(s) = \exp_{p_0}(l W(s))$, and $W(s)$ is a smooth curve in $T_{p_0}M$ by the diffeomorphism property of

<u>Remark 3:</u> The example given in Remark 2 is not the best, since if we require N to be a closed submanifold, it will pass through p_0. A much more persuasive example is the infinite cylinder, with p_0, q_0 on a "diameter" of a slice orthogonal to the vertical axis. Through any point on the vertical line on the cylinder passing through q_0, there are two symmetric minimizing geodesics joining it to p_0. Hence if we are allowed to choose either one of the minimizing geodesics freely for points on N near q_0, there is no way we can make the variation smooth.



cf. Solution to Exercise 6 (c) in this Chapter.

$\exp_{p_0}|_V$. Note that $W(s)$ satisfies $W(0) = \gamma'(0)$ and $|W(s)| = \dfrac{d(p_0, \lambda(s))}{d(p_0, \lambda(0))} = \dfrac{1}{l} d(p_0, \lambda(s))$.

Assume $\gamma|_{(-\varepsilon,\varepsilon)} \subseteq W = \exp_{p_0}(V)$. Let us define a variation of γ by

$$f(s,t) = \exp_{p_0} tW(s), \quad (s,t) \in (-\varepsilon, \varepsilon) \times [0, l].$$

By definition, $f(s,l) = \exp_{p_0}(lW(s)) = \lambda(s) \in N$, $f(s,0) = p_0 = \mathrm{Id}_{p_0}(s)$, and $f(0,t) = \gamma(t)$.

Let V be the variational field of f, defined along γ by $V(t) = \frac{\partial f}{\partial s}\big|_{s=0}$. Then one has $V(0) = \left(\frac{\partial f}{\partial s}\big|_{s=0}\right)\Big|_{t=0} = \frac{\partial \mathrm{Id}_{p_0}}{\partial s}\big|_{s=0} = 0$, $V(l) = \left(\frac{\partial f}{\partial s}\big|_{s=0}\right)\Big|_{t=0} = \frac{\partial \lambda}{\partial s}\big|_{s=0} = v$.

By the first variation formula, we obtain

$$\frac{1}{2}E'(0) = -\int_0^l \left\langle V(t), \frac{D}{dt}\frac{d\gamma}{dt}\right\rangle dt - \left\langle V(0), \frac{d\gamma}{dt}(0)\right\rangle + \left\langle V(l), \frac{d\gamma}{dt}(l)\right\rangle = \left\langle V(l), \frac{d\gamma}{dt}(l)\right\rangle = \langle v, \gamma'(l)\rangle$$

where we used the observation that γ is a geodesic, and that $V(0) = 0$. Moreover, for any $s \in (-\varepsilon, \varepsilon)$, note that

$$E(0) = [\mathrm{length}(\gamma)]^2 / l = [d(p_0,N)]^2 / d(p_0,N) = d(p_0,N) \le d(p_0, \lambda(s)) = d(f(s,0), f(s,l))$$
$$= \mathrm{length}(t \mapsto f(s,t)) \le \mathrm{length}(t \mapsto f(s,t)) \cdot \frac{d(p_0,\lambda(s))}{d(p_0,N)} = [\mathrm{length}(t \mapsto f(s,t))]^2 / l \le E(s),$$

thus $s=0$ is a local minimum of $E(s)$, and hence $E'(0) = 0$. It follows that

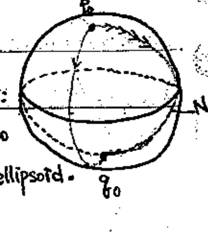
$$0 = \frac{1}{2}E'(0) = \langle v, \gamma'(l)\rangle_{q_0} \quad \text{for all } v \in T_{q_0}N$$

i.e. a minimizing geodesic γ joining p_0 to q_0 is orthogonal to N at q_0, if p_0 and q_0 are not conjugate along γ.

We are now left with the case in which p_0 and q_0 are conjugate along γ. By Proposition 3.5 in Chapter 5 (P117), which says that the conjugate points along γ to q_0 are precisely the critical points of the exponential map $\exp_{q_0}$, and Sard's Theorem, we know there exists some point $p_1 = \gamma(t_1)$ with $0 < t_1 < l$ such that q_0 is not a conjugate point to p_1 along γ (see J. Milnor "Morse Theory" §18, Corollary 18.2; actually we can choose $p_1 \in \gamma$ such that q_0 is not conjugate to p_1 along any geodesic joining p_1 to q_0). Then $\gamma|_{[t_1,l]}$ is a minimizing geodesic joining p_1 to q_0. Note that $d(p_1, q_0) = d(p_1, N)$, for otherwise there exists a piecewise differentiable curve γ' joining p_1 to q_0 such that $\mathrm{length}(\gamma') < \mathrm{length}(\gamma|_{[t_1,l]})$, and thus $(\gamma|_{[0,t_1]}) \cup \gamma'$ is a piecewise differentiable curve joining p_0 to q_0 satisfying $\mathrm{length}((\gamma|_{[0,t_1]}) \cup \gamma') = \mathrm{length}(\gamma|_{[0,t_1]}) + \mathrm{length}(\gamma')$ $< \mathrm{length}(\gamma|_{[0,t_1]}) + \mathrm{length}(\gamma|_{[t_1,l]}) = \mathrm{length}(\gamma) = d(p_0, q_0) = d(p_0, N)$, which is a contradiction. Thus we can apply the previous argument to p_1, q_0 and $\gamma|_{[t_1,l]}$, thus concluding that $\gamma|_{[t_1,l]}$ is orthogonal to N at q_0, i.e. γ is orthogonal to N at q_0. This completes the whole proof.

<u>Remark 1:</u> It is possible that γ may intersect N at other points different from q_0, but there is no guarantee for the orthogonality there.

<u>Remark 2.</u> Why are we worried about p_0, q_0 being conjugate along γ? See this example:



N is the semi-great circle passing through q_0. Let the curve λ be an arc of N through q_0. There is no guarantee that "joining by minimizing geodesics" will produce a smooth variation. Better example may be an ellipsoid.

2. Introduce a complete Riemannian metric on $\mathbb{R}^2$. Prove that $\lim_{r\to\infty}\left(\inf_{x^2+y^2\geq r^2} K(x,y)\right) \leq 0$, where $(x,y)\in\mathbb{R}^2$ and $K(x,y)$ is the Gaussian curvature of the given metric at (x,y).

• Proof: First note that the quantity $A(r) := \inf_{x^2+y^2\geq r^2} K(x,y)$ is monotonically non-decreasing as r increases, thus the limit $\lim_{r\to\infty} A(r)$ does exist, either finitely or equaling $\pm\infty$. If $\lim_{r\to\infty}\left(\inf_{x^2+y^2\geq r^2} K(x,y)\right) = \lim_{r\to\infty} A(r) > 0$, then there exists $\varepsilon_0 > 0$ and $R > 0$ such that $\inf_{x^2+y^2\geq r^2} K(x,y) = A(r) \geq \varepsilon_0 > 0$ for all $r > R$. For simplicity of notations, write

$$M_R := \{(x,y)\in\mathbb{R}^2 : x^2+y^2 > R^2\} \text{ and } B_R := \{(x,y)\in\mathbb{R}^2 : x^2+y^2\leq R^2\}.$$

$\boxed{\text{Step 1:}}$ We claim that B_R is a bounded subset of $\mathbb{R}^2$ under the given metric. Indeed, for any $q\in B_R$, let (W_q, δ_q) be a totally normal neighborhood of q, in which any two points can be joined by a geodesic segment of length δ_q. Then $\{W_q\}_{q\in B_R}$ forms an open cover of $B_R \subset \mathbb{R}^2$. Since B_R is compact in $\mathbb{R}^2$, there exists a finite subcover $\{W_{q_n}\}_{n=1}^N$ out of the open cover $\{W_q\}_{q\in B_R}$. Then $B_R \subset \bigcup_{n=1}^N W_{q_n}$. For simplicity of notations, write (W_n, δ_n) for (W_{q_n}, δ_{q_n}) for all $1\leq n\leq N$. For any point $p\in B_R$, assume $p\in W_k =: V_1$ for some $k\in\{1,\cdots,N\}$. Then for any $q\in V_1$ we have $d(p,q)\leq\delta_k$. Let $V_2 := \bigcup_{W_m\cap V_1\neq\phi} W_m$, then for any $q\in V_2$ we have $d(p,q)\leq d(p,q_1)+d(q_1,q)\leq \delta_k + \max_{1\leq n\leq N}\delta_n \leq 2\cdot\max_{1\leq n\leq N}\delta_n =: 2d$, where q_1 is chosen such that $q_1\in W_m\cap V_1$ for some $1\leq m\leq N$ satisfying $q\in W_m$. Proceeding as this, we can inductively define $V_{n+1} := \bigcup_{W_m\cap V_n\neq\phi} W_m$, then $\{V_n\}$ is an increasing sequence of open subsets of $\mathbb{R}^2$. Since the collection $\{W_m\}_{m=1}^N$ is finite, the collection $\{V_n\}$ stablizes at some sub-index $N_1\in\mathbb{N}$. By the connectedness of B_R in $\mathbb{R}^2$, we have $B_R\subset\bigcup_{n=1}^N W_n = V_{N_1}$, and thus for any $q\in B_R\subset V_{N_1}$ we have $d(p,q)\leq N_1\cdot d$. Note that in this bound N_1 depends on the choice of p but d doesn't. Now for any two points $q_1, q_2\in B_R$, we have from the triangle inequality that $d(q_1,q_2)\leq d(q_1,p)+d(q_2,p)\leq 2N_1 d$, where the bound is uniform for all $q_1,q_2\in B_R$ once we fix $p\in B_R$. This proves our claim that B_R is bounded in $\mathbb{R}^2$ under the given metric. We denote $\mathrm{diam}(B_R)$ for the diameter of B_R under the given metric.

$\boxed{\text{Step 2:}}$ For any $p,q\in M_R$, we claim that any minimizing geodesic joining p,q and lying in M_R must have length no greater than $\pi/\sqrt{\varepsilon_0}$. Indeed, assume γ is such a geodesic, $\gamma\subset M_R$, parametrized by $\gamma:[0,1]\longrightarrow M_R$. Let us consider a parallel vector field $\tilde{V}(t)$ along γ, such that $|\tilde{V}(t)|=1$ for all $t\in[0,1]$, and $\langle\tilde{V}(t),\gamma'(t)\rangle_{\gamma(t)}\equiv 0$ for all $t\in[0,1]$. Assume $\mathrm{length}(\gamma) > \pi/\sqrt{\varepsilon_0}$, we define a vector field along γ given by $V(t) := (\sin\pi t)\tilde{V}(t)$, for all $t\in[0,1]$. It is clear that $V(0)=V(1)=0$, therefore

No.

Date

V generates a proper variation of γ, whose energy we denote by E. Using the formula for the second variation of energy and the fact that V is parallel, we obtain

$$\frac{1}{2}E''(0) = -\int_0^1 \left\langle V, \frac{D^2V}{dt^2} + R\left(\frac{d\gamma}{dt}, V\right)\frac{d\gamma}{dt}\right\rangle dt = -\int_0^1 \left\langle (\sin\pi t)\tilde{V}(t), (-\pi^2\sin\pi t)\tilde{V}(t) + (\sin\pi t)R(\gamma', \tilde{V})\gamma'\right\rangle dt$$

$$= \int_0^1 \left[\pi^2\sin^2\pi t - (\sin^2\pi t)\left\langle R(\gamma'(t), \tilde{V}(t))\gamma'(t), \tilde{V}(t)\right\rangle\right] dt$$

$$= \int_0^1 \left[\pi^2\sin^2\pi t - (\sin^2\pi t)\cdot|\gamma'(t)|^2\cdot K(\gamma(t))\right] dt$$

$$= \int_0^1 \sin^2\pi t\left[\pi^2 - (\mathrm{length}(\gamma))^2\cdot K(\gamma(t))\right] dt$$

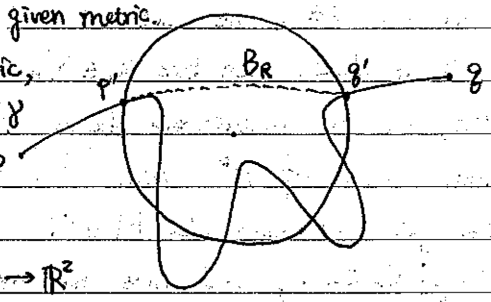
see hand-written remark on P.201 for the reason of this inequality being strict (which is trivial but pedantic)

$$< \int_0^1 \sin^2\pi t\left[\pi^2 - \frac{\pi^2}{\varepsilon_0}\cdot\varepsilon_0\right] dt = 0$$

which by Lemma 2.3 of Chapter 9 (P.194) contradicts the fact that γ is minimizing.

This contradiction proves our claim that any minimizing geodesic lying in M_R entirely which joins $p, q \in M_R$ must have length no greater than $\pi/\sqrt{\varepsilon_0}$.

$\boxed{\text{Step 3:}}$ We claim that $\mathbb{R}^2$ is bounded under the given metric.



In fact, by the completeness of $\mathbb{R}^2$ under this metric, for any $p, q \in \mathbb{R}^2$ there exists a minimizing geodesic γ joining p to q. If $p, q \in B_R$, we have already seen in Step 1 that $d(p,q) \le \mathrm{diam}(B_R) < +\infty$.

If $p, q \in M_R$, let $l := d(p,q)$ and $\gamma : [0, l] \to \mathbb{R}^2$ be a normalized geodesic joining p to q. Since M_R is open, there exists open neighborhoods U, V of p, q respectively such that $\gamma|_{[0,\varepsilon_1)} \subset U \subset M_R$ and $\gamma|_{(\varepsilon_2, l]} \subset V \subset M_R$, for some $\varepsilon_1, \varepsilon_2 \in (0, l)$.

Let $t_1 := \sup\{t \in [0,l] : \gamma|_{[0,t)} \subset M_R\}$ and $t_2 := \inf\{t \in [0,l] : \gamma|_{(t,l]} \subset M_R\}$.

Then t_1, t_2 exists since the sets being considered are both non-empty (containing $\varepsilon_1, \varepsilon_2$ respectively), with $0 < \varepsilon_1 \le t_1 \le t_2 \le \varepsilon_2 < l$. Let $p' := \gamma(t_1)$ and $q' := \gamma(t_2)$. Noting that $\gamma|_{[0,t_1]}$ and $\gamma|_{[t_2,l]}$ are still length-minimizing geodesics in $\mathbb{R}^2$ (for otherwise one may replace $\gamma|_{[0,t_1]}$ and $\gamma|_{[t_2,l]}$ by strictly shorter segments and obtain a strictly shorter piecewise differentiable curve joining p and q, which contradicts the length-minimizing property of γ), and $\gamma|_{[0,t_1)} \subset M_R$, $\gamma|_{(t_2,l]} \subset M_R$, $p' \in B_R$, $q' \in B_R$ (all easy to verify), we know from Step 1 and Step 2 that $d(p,q) \le d(p,p') + d(p',q') + d(q',q)$ $\le \mathrm{length}(\gamma|_{[0,t_1]}) + \mathrm{diam}(B_R) + \mathrm{length}(\gamma|_{[t_2,l]}) \le 2\pi/\sqrt{\varepsilon_0} + \mathrm{diam}(B_R) < +\infty$, where the bound $2\pi/\sqrt{\varepsilon_0} + \mathrm{diam}(B_R)$ is independent of p and q in $\mathbb{R}^2$. If $p \in M_R$, $q \in B_R$ or $p \in B_R$, $q \in M_R$, one similarly argues that $d(p,q) \le \pi/\sqrt{\varepsilon_0} + \mathrm{diam}(B_R) < +\infty$. Thus we proved that $\mathbb{R}^2$ is bounded under the given metric.

$\boxed{\text{Step 4:}}$ Concluding Step 1 through Step 3, we obtain a contradiction as follows: Since $(\mathbb{R}^2, d)$ is complete as a metric space by the Hopf-Rinow Theorem, and $\mathbb{R}^2$ is bounded under this metric, $\mathbb{R}^2$ is compact under this metric by the Hopf-Rinow Theorem (because the completeness of $\mathbb{R}^2$ implies $\mathbb{R}^2$ is closed); however, by Proposition 2.6 in Chapter 7, the topology induced by d on $\mathbb{R}^2$ coincides with the original topology on $\mathbb{R}^2$, which means that $\mathbb{R}^2$ is non-compact. This contradiction tells us that $\lim_{r\to\infty}\left(\inf_{x^2+y^2\ge r^2} K(x,y)\right) \le 0$, as desired.

3. Prove the following generalization of the Theorem of Bonnet-Myers: Let M^n be a compact Riemannian manifold. Suppose that there exist constants $a>0$ and $c \ge 0$ such that for all pairs of points in M^n and for all minimizing geodesics $\gamma(s)$, parametrized by arc length s, joining these points, we have $\mathrm{Ric}(\gamma'(s)) \ge a + \frac{df}{ds}$, along γ, where f is a function of s, satisfying $|f(s)| \le c$ along γ. Then M^n is compact.

Calculate an estimate for the diameter of M^n, and observe that if $f \equiv 0$ and $c=0$, we obtain the Theorem of Bonnet-Myers. The theorem above has application to Relativity, see G. J. Galloway, "A generalization of Myers' Theorem and an application to relativistic cosmology", J. Diff. Geometry, 14 (1979), 105-116.

- Proof: Let $p, q \in M^n$ be two arbitrarily chosen points, and let γ be a normalized minimizing geodesic joining p to q, parametrized by $\gamma: [0,l] \longrightarrow M$. The existence of γ is ensured by the completeness of M. It is enough if we show that l is bounded, because then M is bounded and complete, therefore compact by the Hopf-Rinow Theorem; an estimate for the diameter of M will be found along the way.

Suppose, to the contrary, that $l > \frac{\sqrt{4c^2+\pi^2 a} - 2c}{a}$. Let us consider parallel vector fields $e_1(t), \cdots, e_{n-1}(t)$ along γ which are orthonormal, for each $t \in [0,l]$, and belong to the orthogonal complement of $\gamma'(t)$. Let $e_n(t) = \gamma'(t)$ and let V_j be a vector field along γ given by $V_j(t) = \left(\sin\frac{\pi t}{l}\right) e_j(t)$, $j = 1, \cdots, n-1$. It is clear that $V_j(0) = V_j(1) = 0$, therefore V_j generates a proper variation of γ, whose energy we denote by E_j.

Using the formula for the second variation of energy and the fact that e_j is parallel, we obtain

$$\frac{1}{2}E_j''(0) = -\int_0^\ell \langle V_j,\, V_j'' + R(\gamma', V_j)\gamma' \rangle\, dt = \int_0^\ell \left[\frac{\pi^2}{\ell^2} \sin^2\frac{\pi t}{\ell} - \left(\sin^2\frac{\pi t}{\ell}\right) \cdot \langle e_j(t), R(e_n(t), e_j(t))e_n(t) \rangle \right] dt$$

$$= \int_0^\ell \left(\sin\frac{\pi t}{\ell}\right)^2 \left[\frac{\pi^2}{\ell^2} - R(e_n, e_j, e_n, e_j)(t)\right] dt.$$

Summing on j and using the definition of the Ricci curvature, we get

$$\frac{1}{2}\sum_{j=1}^{n-1} E_j''(0) = (n-1)\int_0^\ell \left(\sin\frac{\pi t}{\ell}\right)^2 \left[\frac{\pi^2}{\ell^2} - \mathrm{Ric}(\gamma'(t))\right] dt \leq (n-1)\int_0^\ell \left(\sin\frac{\pi t}{\ell}\right)^2 \left[\frac{\pi^2}{\ell^2} - a - \frac{df}{dt}\right] dt$$

$$= (n-1)\left[\left(\frac{\pi^2}{\ell^2} - a\right)\int_0^\ell \left(\sin\frac{\pi t}{\ell}\right)^2 dt - \int_0^\ell \frac{df}{dt}\left(\sin\frac{\pi t}{\ell}\right)^2 dt\right]$$

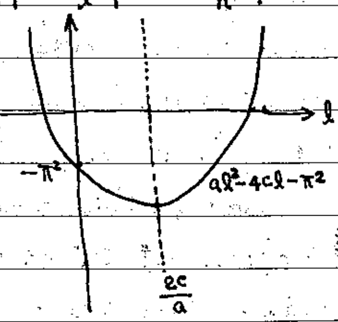
$$= (n-1)\left[\left(\frac{\pi^2}{\ell^2} - a\right)\frac{\ell}{2} - \left(\sin\frac{\pi t}{\ell}\right)^2 f(t)\Big|_{t=0}^{t=\ell} + \int_0^\ell 2\sin\frac{\pi t}{\ell}\cos\frac{\pi t}{\ell}\cdot\frac{\pi}{\ell} f(t)\, dt\right]$$

$$= (n-1)\left[\frac{\pi^2}{2\ell} - \frac{a\ell}{2} + \frac{\pi}{\ell}\int_0^\ell \sin\frac{2\pi t}{\ell}\cdot f(t)\, dt\right] \leq (n-1)\left[\frac{\pi^2}{2\ell} - \frac{a\ell}{2} + \frac{\pi c}{\ell}\int_0^\ell \left|\sin\frac{2\pi t}{\ell}\right| dt\right]$$

$$= (n-1)\left[\frac{\pi^2}{2\ell} - \frac{a\ell}{2} + \frac{\pi c}{\ell}\cdot\frac{2\ell}{\pi}\right] = (n-1)\left(\frac{\pi^2}{2\ell} - \frac{a\ell}{2} + 2c\right)$$

$$= -\frac{n-1}{2\ell}\left(a\ell^2 - 4c\ell - \pi^2\right)$$

where we used elementary calculus $\int_0^\ell \left(\sin\frac{\pi t}{\ell}\right)^2 dt = \frac{\ell}{2}$, $\int_0^\ell \left|\sin\frac{2\pi t}{\ell}\right| dt = \frac{2\ell}{\pi}$.



Noting that $a\ell^2 - 4c\ell - \pi^2 > 0$ for all $\ell > \frac{\sqrt{4c^2 + a\pi^2} - 2c}{a}$, we obtain $\frac{1}{2}\sum_{j=1}^{n-1} E_j''(0) < 0$ for all $\ell > \frac{\sqrt{4c^2 + \pi^2 a} - 2c}{a}$.
As a result there exists an index $j \in \{1, \cdots, n-1\}$ such that $E_j''(0) < 0$, which, by Lemma 2.3 of Chapter 9 (P.194), contradicts the fact that γ is minimizing. This contradiction tells us that $\ell \leq \frac{\sqrt{4c^2 + \pi^2 a} - 2c}{a}$ and thus we obtain an estimate for the diameter of M: $\mathrm{diam}(M) \leq \frac{\sqrt{4c^2 + \pi^2 a} - 2c}{a}$. It is easy to verify that when $f \equiv 0$, $c = 0$ we get back the Bonnet–Myers Theorem.

4. Let M^n be an orientable Riemannian manifold with positive curvature and even dimension. Let γ be a closed geodesic in M, that is, γ is an immersion of the circle S^1 in M that is geodesic at all of its points. Prove that γ is homotopic to a closed curve whose length is strictly less than that of γ.

Hint: The parallel transport along the closed curve γ leaves a vector orthogonal to γ invariant (this comes from the orientability of M and the fact that the dimension is even). Therefore there exists a vector field $V(t)$ parallel along the closed curve γ. Calculate $E_V''(0)$ and show that it is strictly negative. Therefore, close to γ, there exists a closed curve of length smaller than that of γ.

• Proof: $\boxed{1^\circ}$ First we make some observations on the concepts of orientations and parallel transports. This section is supposed to answer the question arising from a first-reading of Weinstein's Theorem (Theorem 3.7 in Chapter 9, P203): Where did we use the assumption that M^n is oriented in the proof? A short answer would be: In an oriented Riemannian manifold the parallel transport is orientation-preserving.

What is an orientation? A smooth manifold M is orientable if and only if there exists a non-vanishing C^∞ n-form, where $n = \dim M$; such an n-form ω on an orientable manifold M is called an orientation of M. This definition is techniquely very convenient as well as intuitive: for $p \in M$, a basis $\{e_i(p)\}_{i=1}^n$ is positively oriented if and only if $\omega(e_1, \cdots, e_n) > 0$, and negatively oriented otherwise. It is known from linear algebra easily that the transformation matrix between two positively oriented bases has positive determinant.

Keeping this equivalent definition of orientability in mind, let $P_{\gamma(t)}: T_{\gamma(0)}M \longrightarrow T_{\gamma(t)}M$ denote the parallel transport operator along an arbitrary piecewise differentiable curve $\gamma: [0, t] \longrightarrow M$. If $\{e_i\}_{i=1}^n$ is a positively oriented basis of $T_{\gamma(0)}M$, then $\{P_{\gamma(t)} e_i\}_{i=1}^n$ must also be a positively oriented basis of $T_{\gamma(t)}M$. Indeed, the linear independence of $\{P_{\gamma(t)} e_i\}_{i=1}^n$ follows from the "isometry" nature of parallel transports (it is not really an isometry since according to the definition on P38 an isometry is required to be a diffeomorphism at the manifold level in the first place); the global non-vanishing nature of ω then ensures that the smooth mapping $t \mapsto \omega(P_{\gamma(t)} e_1, \cdots, P_{\gamma(t)} e_n)$ is non-vanishing for all t on which γ is defined. Since $\omega(P_{\gamma(0)} e_1, \cdots, P_{\gamma(0)} e_n) = \omega(e_1, \cdots, e_n) > 0$, we must have $\omega(P_{\gamma(t)} e_1, \cdots, P_{\gamma(t)} e_n) > 0$ by the connectedness of γ. This proves that $\{P_{\gamma(t)} e_i\}_{i=1}^n$ is also a positively oriented basis for $T_{\gamma(t)}M$. So far we have been following Exercise 1 of Chapter 2 (PP.56) closely, and it should be clear now why we need the Riemannian manifold to be oriented in Weinstein's Theorem.

$\boxed{2^\circ}$ Let us now turn to our problem. Note that $\gamma : [0,l] \longrightarrow M$ is a closed geodesic, i.e. $\gamma(0) = \gamma(l) =: p \in M$, and $\gamma'(0) = \gamma'(l) =: v \in T_pM$ since γ is smooth at p. For simplicity of notations, let $P := P_{\gamma(l)} : T_pM = T_{\gamma(0)}M \longrightarrow T_{\gamma(l)}M = T_pM$. By 1° we know that P is orientation-preserving, thus $\det P = 1$ (since P preserves inner products at p gives $PP^* = P^*P = \mathrm{Id} \Rightarrow \det P = \pm 1$ in the first place). Moreover, let $N_v \subset T_pM$ denote the orthogonal complement of $v = \gamma'(0) = \gamma'(l) \in T_pM$, then for any $w \in N_v$ we have

$$0 = \langle v, w \rangle_{p=\gamma(0)} = \langle P(\gamma'(0)), Pw \rangle_{\gamma(l)=p} = \langle \gamma'(l), Pw \rangle_p = \langle \gamma'(0), Pw \rangle_p = \langle v, Pw \rangle$$

where we used that γ is a geodesic and the uniqueness of parallel transports along a given curve ($\gamma'(l)$ is the unique parallel transport of $\gamma'(0)$ along γ). Hence $P|_{N_v} : N_v \to N_v$ is an orthogonal transformation on N_v. Since M is even dimensional, N_v is odd dimensional, and hence $P|_{N_v}$ has 1 as its eigenvalue. Let $u \in N_v \subset T_pM$ be the corresponding eigenvector of $P|_{N_v}$. Then one has $Pu = P|_{N_v}(u) = u \in N_v$, and hence one can parallel transport u along γ and obtain a well-defined vector field $V(t) := P_{\gamma(t)}u$ parallel along the closed geodesic γ. By Proposition 2.2, there exists a variation $f : (-\varepsilon, \varepsilon) \times [0,l] \to M$ of γ such that V is the variational field of f. Denote E for the energy associated to this variation f. By the formula for the second variation,

$$\frac{1}{2}E''(0) = -\int_0^l \left\langle V(t), \cancelto{0}{\frac{D^2V}{dt^2}(t)} + R(\gamma'(t), V(t))\gamma'(t) \right\rangle dt - \cancel{\left\langle \frac{D}{ds}\frac{\partial f}{\partial s}(0,0), \gamma'(0) \right\rangle_{\gamma(0)}} + \cancel{\left\langle \frac{D}{ds}\frac{\partial f}{\partial s}(0,l), \gamma'(l) \right\rangle_{\gamma(l)}}$$

$$- \left\langle V(0), \cancelto{0}{\frac{DV}{dt}(0)} \right\rangle_{\gamma(0)} + \left\langle V(l), \cancelto{0}{\frac{DV}{dt}(l)} \right\rangle_{\gamma(l)}$$

$$= -\int_0^l \langle R(\gamma', V)\gamma', V \rangle(t)\, dt < 0 \quad \text{since } M \text{ has positive curvature.}$$

Here we used the construction that V is parallel along γ, and the fact that γ is closed, which gives $\gamma'(0) = \gamma'(l)$ and $f(s,0) = f(s,l)$ for all $s \in (-\varepsilon, \varepsilon)$. Therefore, there exists a curve c in the variation such that $(l(c))^2 \le l \cdot E(c) < l \cdot E(\gamma) = (l(\gamma))^2$, i.e. length$(c)$ < length(γ). By the construction of the variation f, we know that c is also closed (parametrized by $t \mapsto f(s_0, t)$ for some fixed $s_0 \in (-\varepsilon, \varepsilon)$), and that γ is homotopic to c via the obvious (smooth) homotopy $f : [0, s_0] \times [0,l] \longrightarrow M$ (if $s_0 > 0$) or $f : [s_0, 0] \times [0,l] \longrightarrow M$ (if $s_0 < 0$).

Remark: This problem is a good example illustrating the phenomenon that a closed geodesic being not length-minimizing in particular situations. This is a stupid remark since closed geodesics are geometrically so obviously different from geodesic segments.

5. Let N_1 and N_2 be two closed disjoint submanifolds of a compact Riemannian manifold.

a) Show that the distance between N_1 and N_2 is assumed by a geodesic γ perpendicular to both N_1 and N_2.

b) Show that, for any orthogonal variation $f(t,s)$ of γ, with $f(0,s)\in N_1$ and $f(\ell,s)\in N_2$, we have the following expression for the formula for the second variation

$\frac{1}{2}E''(0) = I_\ell(V,V) + \langle V(\ell), S^{(2)}_{\gamma'(\ell)}V(\ell)\rangle - \langle V(0), S^{(1)}_{\gamma'(0)}V(0)\rangle$

where V is the variational vector field and $S^{(i)}_{\gamma'}$ is the linear map associated to the second fundamental form of N_i in the direction of γ', $i=1,2$.

• $\boxed{a)}$ By Hopf-Rinow Theorem, the compact Riemannian manifold, denoted as M, is complete as a metric space. Since N_1, N_2 are both closed subspaces of (M,d) (cf. Proposition 2.6 in Chapter 7) and $N_1\cap N_2=\phi$, by standard functional analysis we have $d(N_1,N_2)=\inf_{q_1\in N_1, q_2\in N_2} d(q_1,q_2)>0$. Indeed, there exists sequences $\{q_1^{(n)}\}_{n=1}^\infty\subset N_1$, $\{q_2^{(n)}\}_{n=1}^\infty\subset N_2$ such that $d(q_1^{(n)},q_2^{(n)})\longrightarrow d(N_1,N_2)$ as $n\to\infty$. Since M is compact and N_1,N_2 are both closed in M, we know N_1,N_2 are both compact. Passing to subsequences if necessary, we have $q_1^{(n)}\longrightarrow p_1\in N_1$, $q_2^{(n)}\longrightarrow p_2\in N_2$ as $n\to\infty$; in particular, $\{q_1^{(n)}\}_{n=1}^\infty$ and $\{q_2^{(n)}\}_{n=1}^\infty$ are both Cauchy sequences. Then by triangle inequality we have

$|d(p_1,p_2)-d(N_1,N_2)|\le |d(p_1,p_2)-d(p_1,q_2^{(n)})|+|d(p_1,q_2^{(n)})-d(q_1^{(n)},q_2^{(n)})|+|d(q_1^{(n)},q_2^{(n)})-d(N_1,N_2)|$

$\le d(p_2,q_2^{(n)})+d(p_1,q_1^{(n)})+|d(q_1^{(n)},q_2^{(n)})-d(N_1,N_2)|\longrightarrow 0$ as $n\to\infty$

i.e. $d(p_1,p_2)=d(N_1,N_2)$. If $d(N_1,N_2)=0$, then $d(p_1,p_2)=0\Rightarrow p_1=p_2\in N_1\cap N_2$, Contradicting our assumption that $N_1\cap N_2=\phi$.

Note that $d(p_1,N_2)\le d(p_1,p_2)=d(N_1,N_2)\le d(p_1,N_2)$, thus $d(p_1,p_2)=d(p_1,N_2)$. By Exercise 1 in this Chapter, a minimizing geodesic γ joining p_1 to p_2 (whose existence follows from the geodesic completeness of M) is orthogonal to N_2 at p_2. Similarly, $d(p_1,p_2)=d(N_1,p_2)$, and thus γ is orthogonal to N_1 at p_1 for the same reason. Moreover, since γ is minimizing, there holds $\mathrm{length}(\gamma)=d(p_1,p_2)=d(N_1,N_2)$.

$\boxed{b)}$ Recall the formula for the second variation

$\frac{1}{2}E''(0)=I_\ell(V,V)-\langle\frac{D}{ds}\frac{\partial f}{\partial s},\frac{\partial f}{\partial t}\rangle(0,0)+\langle\frac{D}{ds}\frac{\partial f}{\partial s},\frac{\partial f}{\partial t}\rangle(0,\ell)$

where $I_\ell(V,V)=\int_0^\ell\left[\langle V',V'\rangle-\langle R(\gamma',V)\gamma',V\rangle\right]dt$ is the index form.

Note that $s\mapsto f(\ell,s)$ is a curve in N_2. Let $\nabla,\nabla^{(1)},\nabla^{(2)}$ denote the Levi-Civita connection of M, N_1, N_2 respectively. Recall the formulae $\nabla_X Y=\nabla_X^{(i)}Y+B^{(i)}(X,Y)$,

No.

Date

$\nabla_X \eta = -S^{(j)}_\eta(X) + \nabla^{(j)\perp}_X \eta$, and $\langle B^{(j)}(X,Y), \eta\rangle = \langle S^{(j)}_\eta(X), Y\rangle$ for $j=1,2$. We have

$$\frac{D}{ds}\frac{\partial h}{\partial s} = \nabla_{\frac{\partial h}{\partial s}}\frac{\partial h}{\partial s} = \nabla^{(2)}_{\frac{\partial h}{\partial s}}\frac{\partial h}{\partial s} + B^{(2)}\left(\frac{\partial h}{\partial s}, \frac{\partial h}{\partial s}\right)$$

and $\gamma'(l)$ being a normal vector of N_2 at P_2 since γ is orthogonal to N_2 at P_2. It follows that

$$\left\langle \frac{D}{ds}\frac{\partial h}{\partial s}, \frac{\partial h}{\partial t}\right\rangle(0,l) = \left\langle \frac{D}{ds}\frac{\partial h}{\partial s}(0,l), \gamma'(l)\right\rangle = \left\langle B^{(2)}\left(\frac{\partial h}{\partial s}, \frac{\partial h}{\partial s}\right)(0,l), \gamma'(l)\right\rangle$$

$$= \left\langle S^{(2)}_{\gamma'(l)}\left(\frac{\partial h}{\partial s}(0,l)\right), \frac{\partial h}{\partial s}(0,l)\right\rangle = \left\langle S^{(2)}_{\gamma'(l)}V(l), V(l)\right\rangle.$$

Similarly, one has $\left\langle \frac{D}{ds}\frac{\partial h}{\partial s}, \frac{\partial h}{\partial t}\right\rangle(0,0) = \left\langle S^{(1)}_{\gamma'(0)}V(0), V(0)\right\rangle$.

Therefore, we have the following expression for the formula for the second variation

$$\frac{1}{2}E''(0) = I_l(V,V) + \left\langle \frac{D}{ds}\frac{\partial h}{\partial s}, \frac{\partial h}{\partial t}\right\rangle(0,l) - \left\langle \frac{D}{ds}\frac{\partial h}{\partial s}, \frac{\partial h}{\partial t}\right\rangle(0,0)$$

$$= I_l(V,V) + \left\langle V(l), S^{(2)}_{\gamma'(l)}V(l)\right\rangle - \left\langle V(0), S^{(1)}_{\gamma'(0)}V(0)\right\rangle.$$

6. Let $\tilde{M}$ be a complete simply connected Riemannian manifold, with curvature $K \leq 0$. Let $\gamma : (-\infty, \infty) \longrightarrow \tilde{M}$ be a normalized geodesic and let $p \in \tilde{M}$ be a point which does not belong to γ. Let $d(s) = d(p, \gamma(s))$.

a) Consider the minimizing geodesic $\sigma_s : [0, d(s)] \longrightarrow \tilde{M}$ joining p to $\gamma(s)$, that is, $\sigma_s(0) = p$, $\sigma_s(d(s)) = \gamma(s)$. Consider the variation $h(t,s) = \sigma_s(t)$, and show that:

(i) $\frac{1}{2}E'(s) = \langle \gamma'(s), \sigma_s'(d(s)) \rangle$,

(ii) $\frac{1}{2}E''(s) > 0$.

b) Conclude from (i) that s_0 is a critical point of d if and only if $\langle \gamma'(s_0), \sigma_{s_0}'(d(s_0)) \rangle = 0$. Conclude from (ii) that d has a unique critical point, which is a minimum.

c) From (b), it follows that if $\tilde{M}$ is complete, simply connected and has curvature $K \leq 0$, then a point off the geodesic γ of $\tilde{M}$ can be connected by a unique minimizing geodesic perpendicular to γ. Show by examples that the condition on the curvature and the condition of simple connectivity are essential to the theorem.

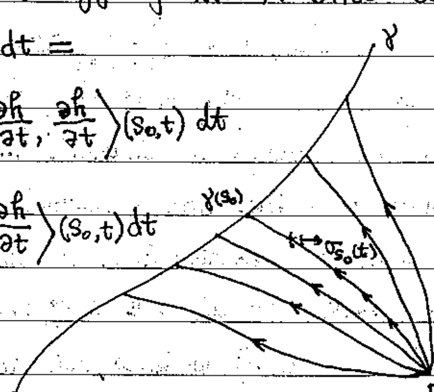
• $\boxed{a)}$ Let us fix some $s_0 \in \mathbb{R}$ and consider the geodesic $\sigma_{s_0} : [0, d(s_0)] \longrightarrow \tilde{M}$ joining p to $\gamma(s_0)$. The existence of such a geodesic comes from the completeness of $\tilde{M}$. By Lemma 3.2 in Chapter 7 (P.149), $\exp_p : T_p\tilde{M} \longrightarrow \tilde{M}$ is a local diffeomorphism, thus there exists a neighborhood V of $d(s_0)\sigma_{s_0}'(0) \in T_p\tilde{M}$ and a neighborhood W of $\gamma(s_0) = \exp_p(d(s_0)\sigma_{s_0}'(0))$ such that $\exp_p|_V : V \longrightarrow W$ is a diffeomorphism. In particular, for any $\gamma(s) \in \gamma \cap W$, there exists $W(s) \in V$ such that $\gamma(s) = \exp_p(d(s_0)W(s))$, and $W(s)$ is a smooth curve in T_pM with $W(s_0) = \sigma_{s_0}'(0)$. Thus a local variation of σ_{s_0} via geodesics exists. Without loss of generality, we may parametrize this variation by $h(s,t) : [s_0-\varepsilon, s_0+\varepsilon] \times [0, d(s_0)] \longrightarrow \tilde{M}$, where $h(0,t) = \sigma_{s_0}(t)$. Let $E(s)$ denote for the energy of $h(s,t)$. Direct computation gives $\frac{1}{2}E'(s_0) = \frac{1}{2}\frac{\partial}{\partial s}\Big|_{s=s_0} \int_0^{d(s_0)} \left\langle \frac{\partial h}{\partial t}, \frac{\partial h}{\partial t} \right\rangle (s,t)\, dt =$

$$= \int_0^{d(s_0)} \frac{\partial}{\partial s}\Big|_{s=s_0} \left\langle \frac{\partial h}{\partial t}, \frac{\partial h}{\partial t} \right\rangle (s,t)\, dt = \int_0^{d(s_0)} \left\langle \frac{D}{ds}\frac{\partial h}{\partial t}, \frac{\partial h}{\partial t} \right\rangle (s_0,t)\, dt$$

$$= \int_0^{d(s_0)} \left\langle \frac{D}{dt}\frac{\partial h}{\partial s}, \frac{\partial h}{\partial t} \right\rangle (s_0,t)\, dt = \int_0^{d(s_0)} \frac{\partial}{\partial t} \left\langle \frac{\partial h}{\partial s}, \frac{\partial h}{\partial t} \right\rangle (s_0,t)\, dt$$

$$= \left\langle \frac{\partial h}{\partial s}, \frac{\partial h}{\partial t} \right\rangle (s_0, d(s_0)) - \left\langle \frac{\partial h}{\partial s}, \frac{\partial h}{\partial t} \right\rangle (s_0, 0)$$

$$= \left\langle \frac{\partial h}{\partial s}(s_0, d(s_0)), \frac{\partial h}{\partial t}(s_0, d(s_0)) \right\rangle = \left\langle \gamma'(s_0), \sigma_{s_0}'(d(s_0)) \right\rangle$$



Replacing s_0 for arbitrary $s \in \mathbb{R}$, we established (i).

Either by direct computation or by directly applying the formula for the second variation, we have

$$\frac{1}{2}E''(s_0) = \frac{\partial}{\partial s}\Big|_{s=s_0}\left(\frac{1}{2}E'(s)\right) = \frac{\partial}{\partial s}\Big|_{s=s_0}\int_0^{d(s_0)}\left\langle \frac{D}{ds}\frac{\partial h}{\partial t}, \frac{\partial h}{\partial t}\right\rangle(s,t)\,dt$$

$$= \int_0^{d(s_0)} \frac{\partial}{\partial s}\Big|_{s=s_0}\left\langle \frac{D}{ds}\frac{\partial h}{\partial t}, \frac{\partial h}{\partial t}\right\rangle(s,t)\,dt$$

$$= \int_0^{d(s_0)}\left\langle \frac{D}{ds}\frac{D}{ds}\frac{\partial h}{\partial t}, \frac{\partial h}{\partial t}\right\rangle(s_0,t)\,dt + \int_0^{d(s_0)}\left\langle \frac{D}{ds}\frac{\partial h}{\partial t}, \frac{D}{ds}\frac{\partial h}{\partial t}\right\rangle(s_0,t)\,dt$$

$$= \int_0^{d(s_0)}\left\langle \frac{D}{ds}\frac{D}{dt}\frac{\partial h}{\partial s}, \frac{\partial h}{\partial t}\right\rangle(s_0,t)\,dt + \int_0^{d(s_0)}\left|\frac{D}{ds}\frac{\partial h}{\partial t}\right|^2(s_0,t)\,dt$$

$$= \int_0^{d(s_0)}\left\langle \frac{D}{dt}\frac{D}{ds}\frac{\partial h}{\partial s} - R\left(\frac{\partial h}{\partial s}, \frac{\partial h}{\partial t}\right)\frac{\partial h}{\partial s}, \frac{\partial h}{\partial t}\right\rangle(s_0,t)\,dt + \int_0^{d(s_0)}\left|\frac{D}{ds}\frac{\partial h}{\partial t}\right|^2(s_0,t)\,dt$$

$$= \int_0^{d(s_0)}\left\langle \frac{D}{dt}\frac{D}{ds}\frac{\partial h}{\partial s}, \frac{\partial h}{\partial t}\right\rangle(s_0,t)\,dt - \int_0^{d(s_0)}\left\langle R\left(\frac{\partial h}{\partial s}, \frac{\partial h}{\partial t}\right)\frac{\partial h}{\partial s}, \frac{\partial h}{\partial t}\right\rangle(s_0,t)\,dt + \int_0^{d(s_0)}\left|\frac{D}{ds}\frac{\partial h}{\partial t}\right|^2(s_0,t)\,dt$$

$$= \int_0^{d(s_0)}\frac{\partial}{\partial t}\left\langle \frac{D}{ds}\frac{\partial h}{\partial s}, \frac{\partial h}{\partial t}\right\rangle(s_0,t)\,dt - \int_0^{d(s_0)} R\left(\frac{\partial h}{\partial s}, \frac{\partial h}{\partial t}, \frac{\partial h}{\partial s}, \frac{\partial h}{\partial t}\right)(s_0,t)\,dt + \int_0^{d(s_0)}\left|\frac{D}{ds}\frac{\partial h}{\partial t}\right|^2(s_0,t)\,dt$$

$$= \left\langle \frac{D}{ds}\frac{\partial h}{\partial s}, \frac{\partial h}{\partial t}\right\rangle(s_0,d(s_0)) - \left\langle \frac{D}{ds}\frac{\partial h}{\partial s}, \frac{\partial h}{\partial t}\right\rangle(s_0,0) - \int_0^{d(s_0)} K\left(\frac{\partial h}{\partial s}, \frac{\partial h}{\partial t}\right)(s_0,t)\,dt + \int_0^{d(s_0)}\left|\frac{D}{ds}\frac{\partial h}{\partial t}\right|^2(s_0,t)\,dt$$

$$= \left\langle \cancelto{0}{\frac{D}{ds}\Big|_{s=s_0}\gamma'(s)}, \frac{\partial h}{\partial t}(s_0,d(s_0))\right\rangle - \left\langle 0, \sigma_{s_0}'(0)\right\rangle - \int_0^{d(s_0)} K\left(\frac{\partial h}{\partial s}(s_0,t), \frac{\partial h}{\partial t}(s_0,t)\right)dt + \int_0^{d(s_0)}\left|\frac{D}{ds}\frac{\partial h}{\partial t}\right|^2(s_0,t)\,dt$$

$$= -\int_0^{d(s_0)} K\left(\frac{\partial h}{\partial s}(s_0,t), \frac{\partial h}{\partial t}(s_0,t)\right)dt + \int_0^{d(s_0)}\left|\frac{D}{ds}\frac{\partial h}{\partial t}(s_0,t)\right|^2 dt$$

$$\geq \int_0^{d(s_0)}\left|\frac{D}{ds}\frac{\partial h}{\partial t}(s_0,t)\right|^2 dt > 0,$$ for otherwise $\frac{D}{ds}\frac{\partial h}{\partial t}(s_0,t) \equiv 0$, $\forall\, t\in[0,d(s_0)]$, in particular $\frac{D}{ds}\frac{\partial h}{\partial t}(s_0,0) = 0$.

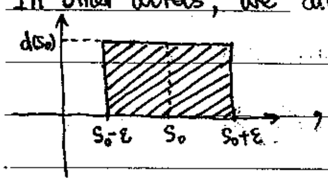
If this is the case, then $\frac{D}{dt}\frac{\partial h}{\partial s}(s_0,0) = \frac{D}{ds}\frac{\partial h}{\partial t}(s_0,0) = 0$.

By Exercise 2 in Chapter 5 (p.119), $V(t) := \frac{\partial h}{\partial s}(s_0,t)$ is a Jacobi field, and the above observation gives in this case $\frac{DV}{dt}(0) = 0$, $V(0) = \frac{\partial}{\partial s}h(s_0,0) = \frac{\partial}{\partial s}\Big|_{s=s_0} Id_p = 0$. By the uniqueness of Jacobi fields along a geodesic with prescribed initial datum, this implies $V(t) \equiv 0$ for all $t\in[0,d(s_0)]$.
In this case, the variation h constructed above is locally trivial around $t \mapsto \sigma_{s_0}(t)$, i.e. $h(s,t) = h(s_0,t)$ for all $s\in(s_0-\eta, s_0+\eta)$ for some sufficiently small $\eta>0$, since for any fixed $t^*\in[0,d(s_0)]$ we know that $s\mapsto h(s,t^*)$ is the local flow around $h(s_0,t^*)$ of the vector field $s\mapsto \frac{\partial h}{\partial s}(s,t^*)$, and it follows from the uniqueness of the local flow with initial conditions $h(s_0,t^*) = h(s_0,t^*)$, $\frac{\partial h}{\partial s}(s_0,t^*) = V(t^*) = 0$ that $h(s,t) = h(s_0,t)$ for all $s\in(s_0-\eta, s_0+\eta)$.

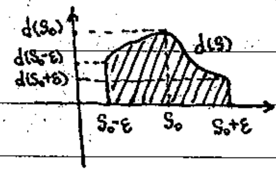
This contradicts our construction of $h(s,t)$, since in particular $h(s, d(s_0)) = h(s_0, d(s_0))$ for all $s \in (s_0-\eta, s_0+\eta)$, but we constructed $h(s,t) = \exp_p(tW(s))$, which then yields $h(s, d(s_0)) = \exp_p(d(s_0)W(s)) = \gamma(s)$ and hence $\gamma(s) = \gamma(s_0)$ for all $s \in (s_0-\eta, s_0+\eta)$, contradicting our assumption that γ is a normalized geodesic and hence $|\gamma'(s)| \equiv 1$ for all $s \in \mathbb{R}$. This establishes (ii).

$\boxed{b)}$ Note that in our setting $t \mapsto h(s_0,t) = \sigma_{s_0}(t)$ is a geodesic, hence $E(s_0) = \frac{(L(s_0))^2}{d(s_0)} = d(s_0)$, for all $s_0 \in (-\infty, \infty)$. Since $d(s_0) = \text{length}(\sigma_{s_0})$ is independent of the parametrization of σ_{s_0}, $E(s_0)$ is also independent of the parametrization of σ_{s_0}, and hence $E(s_0) = \int_0^{d(s_0)} \left\langle \frac{\partial h}{\partial t}, \frac{\partial h}{\partial t} \right\rangle (s_0,t)\, dt$ is independent of the choice of variations of σ_{s_0}. (Note here that $E(s_0)$ is understood as $\left[\int_0^{d(s_0)} \left\langle \frac{\partial h}{\partial t}, \frac{\partial h}{\partial t} \right\rangle (s,t)\, dt\right]\Big|_{s=s_0}$, not $\left[\int_0^{d(s)} \left\langle \frac{\partial h}{\partial t}, \frac{\partial h}{\partial t} \right\rangle (s,t)\, dt\right]\Big|_{s=s_0}$.

In other words, we always first fix an $s_0 \in \mathbb{R}$, then take a variation defined on



, not a variation defined on



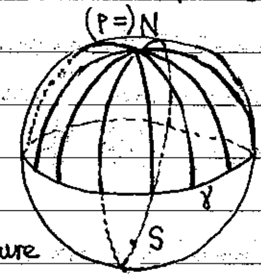
!)

In other words, in our situation, $E(s)$, as well as $L(s) = d(s)$, is a well-defined function of $s \in \mathbb{R}$. Now if s_0 is a critical point of d if and only if s_0 is a critical point of E (since $E(s) = d(s)$), which is equivalent by (a)(i) to the equality
$$\langle \gamma'(s_0), \sigma_{s_0}'(d(s_0)) \rangle = \tfrac{1}{2}E'(s_0) = 0.$$
Moreover, it follows from (a)(ii) that $\frac{1}{2}E''(s_0) > 0$, which means that s_0 must be a local strict minimum. In particular, this also implies that s_0 is a locally unique minimum of d. To see that d has a unique critical point, we shall make use of our simply-connectivity assumption of $\tilde{M}$ (note this is the first time we use simply-connectivity so far!). Indeed, in a) we constructed the variation $h(s,t)$ locally around $\sigma_{s_0}(t)$ via the local diffeomorphism nature of $\exp_p : T_p\tilde{M} \to \tilde{M}$; moreover, when $\tilde{M}$ is in addition simply connected, we conclude from the Cartan-Hadamard Theorem that $\exp_p : T_p\tilde{M} \to \tilde{M}$ is a global diffeomorphism, and hence $h(s,t)$ can be defined on the entire $(-\infty,\infty) \times [0, d(s_0)]$, since $\gamma : (-\infty,\infty) \to \tilde{M}$ is normalized and defined on the entire $(-\infty, \infty)$. By (a)(ii), $E(s)$ is a strictly convex smooth function defined on the entire $\mathbb{R}$. Recall that we learnt in kindergarten that a strictly convex function has a unique minimum on its connected domain of definition, therefore

$d(s) = E(s)$ has a globally unique minimum on $\mathbb{R}$. This proves that d has a unique critical point which is a minimum.

$\boxed{c)}$ If $\tilde{M}$ is complete, simply connected with non-positive sectional curvature, then it follows from b) that a point off the geodesic γ of $\tilde{M}$ can be connected by a unique minimizing geodesic satisfying $d'(s) = E'(s) = 0$, or equivalently (by (a)(i)) speaking, satisfying $\langle \gamma'(s), \sigma_s'(d(s)) \rangle = 0$, i.e. perpendicular to γ.

$\boxed{1°}$ Necessity of the condition on the curvature: Consider the counterexample of a unit sphere S^2 in $\mathbb{R}^3$:



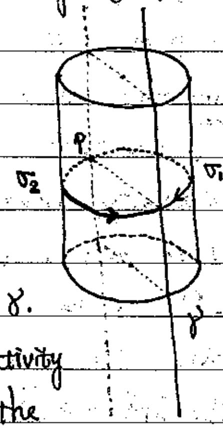
There are infinitely many minimizing geodesics connecting the north pole to the equator which are perpendicular to the equator.

The problem is because of the positive curvature one can not conclude in (a)(ii) that $\frac{1}{2}E''(s) > 0$: the curvature term in

$$\frac{1}{2}E''(s) = -\int_0^{d(s)} K\left(\frac{\partial h}{\partial s}(s,t), \frac{\partial h}{\partial t}(s,t)\right) dt + \int_0^{d(s)} \left|\frac{D}{ds}\frac{\partial h}{\partial t}(s,t)\right|^2 dt$$

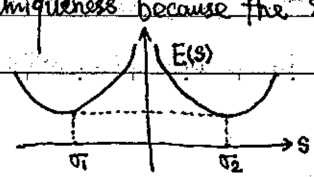
can make the whole expression on the right hand side non-positive. In fact, for the counterexample above, one has $\frac{1}{2}E''(s) \equiv 0$ and thus $\langle \gamma'(s), \sigma_s'(d(s)) \rangle = \frac{1}{2}E'(s) \equiv 0$ for all $s \in (-\infty, \infty)$.

$\boxed{2°}$ Necessity of the condition of simple connectivity: consider the two dimensional infinite cylinder in $\mathbb{R}^3$:



If γ is a geodesic on the cylinder parallel to the vertical axis, and p is a point off γ such that the plane determined by p and γ is a symmetric plane of the cylinder, then there are two minimizing geodesics connecting p to γ which are perpendicular to γ.

The problem is because of the lack of simple connectivity the smooth function $d(s) = E(s)$ is not defined on the entire $\mathbb{R}$, although (a) is still valid for this example and $E(s)$ is actually strictly convex. Note that σ_1, σ_2 are both locally unique minima of $E(s)$, but one can not pass from local uniqueness to global uniqueness because the strictly convex smooth function $E(s)$ is not globally defined.



Remark 1: We know that in a Euclidean space a point off a straight line γ is connected uniquely by a segment (of a straight line) perpendicular to γ. This problem draws a further similarity shared by the Euclidean space and a complete simply connected Riemannian manifold with non-positive curvature, apart from (or based on ?) the renowned Cartan-Hadamard Theorem.

Remark 2: The geometry is (almost) always independent of parametrizations: the fact in c) has nothing to do with the parametrization of γ. The requirement that γ being normalized is a typical technical choice of parametrizations; it is supposed to facilitate rather than obscure the line of proof.

Remark 3: It is not likely that one can use the variation $h(s,t) = \sigma_s(t)$ suggested in (a) to work out everything. For that variation, every geodesic $t \longmapsto \sigma_s(t)$ is normalized with unit speed, and $E(s) = \int_0^{d(s)} \left\langle \frac{\partial h}{\partial t}, \frac{\partial h}{\partial t} \right\rangle (s,t)\, dt = \int_0^{d(s)} dt = d(s) = L(s)$, and $E'(s) = L'(s)$. But one can not do anything with $L'(s)$ for the same reason: $L(s) = \int_0^{d(s)} \sqrt{\left\langle \frac{\partial h}{\partial t}, \frac{\partial h}{\partial t} \right\rangle (s,t)}\, dt = \int_0^{d(s)} dt = d(s)$, and differentiating $L(s)$ does not provide us with any useful information:

$$L'(s) = \left\langle \frac{\partial h}{\partial t}, \frac{\partial h}{\partial t} \right\rangle (s, d(s)) \cdot d'(s) + \int_0^{d(s)} \frac{\partial}{\partial s} \sqrt{\left\langle \frac{\partial h}{\partial t}, \frac{\partial h}{\partial t} \right\rangle (s,t)}\, dt$$

$$= 1 \cdot d'(s) + \int_0^{d(s)} 0\, dt = d'(s) = L'(s)$$

This tells us that the essence of energy variation is to carry out a change of variable so as to transform the information of length into information revealed by the behavior of speed vectors; if we normalize the speed vector instead, all the information remain in the length (or energy) undisclosed. This contemplation teaches us why we always want to do variation on a rectangle domain rather than a domain of more flexible shape (cf. Solution to b), a remark inline illustrating the shape of the domain on which a variation is defined): we want to squeeze information out of the domain of definition!

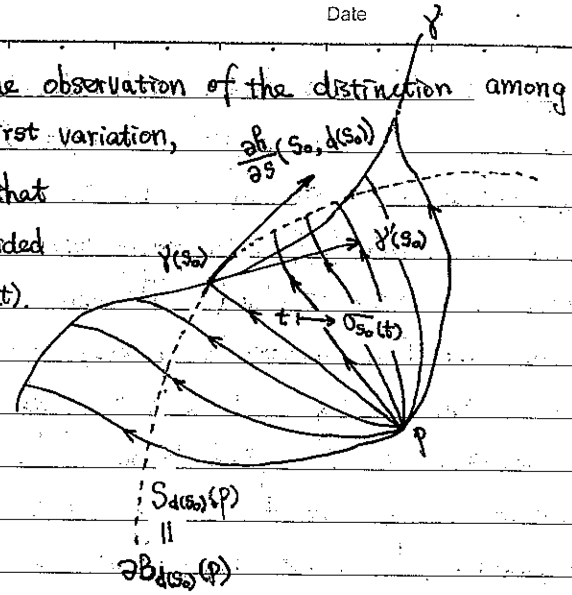
Here we remark that there seems to be a paradox here: if we define $h(s,t) = \sigma_s(t)$, then $0 = \frac{\partial}{\partial s}\Big|_{s=s_0} \left\langle \frac{\partial h}{\partial t}, \frac{\partial h}{\partial t} \right\rangle (s,t) = 2 \left\langle \frac{D}{ds}\Big|_{s=s_0} \frac{\partial h}{\partial t}(s,t), \frac{\partial h}{\partial t}(s,t) \right\rangle = 2 \left\langle \frac{D}{ds}\frac{\partial h}{\partial t}(s_0,t), \frac{\partial h}{\partial t}(s_0,t) \right\rangle$

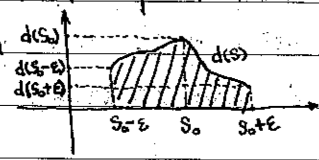
$$= 2\left\langle \frac{D}{dt}\frac{\partial h}{\partial s}(s_0,t), \frac{\partial h}{\partial t}(s_0,t) \right\rangle = 2\frac{d}{dt}\left\langle \frac{\partial h}{\partial s}(s_0,t), \frac{\partial h}{\partial t}(s_0,t) \right\rangle - 2\left\langle \frac{\partial h}{\partial s}(s_0,t), \frac{D}{dt}\frac{\partial h}{\partial t}(s_0,t) \right\rangle$$

$$= 2\frac{d}{dt}\left\langle \frac{\partial h}{\partial s}(s_0,t), \frac{\partial h}{\partial t}(s_0,t) \right\rangle \Rightarrow \left\langle \frac{\partial h}{\partial s}(s_0,d(s_0)), \frac{\partial h}{\partial t}(s_0,d(s_0)) \right\rangle = \left\langle \frac{\partial h}{\partial s}(s_0,0), \frac{\partial h}{\partial t}(s_0,t) \right\rangle = 0$$

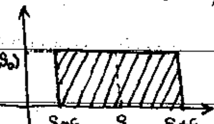
since $\frac{\partial h}{\partial s}(s_0,0) = \frac{\partial}{\partial s}\Big|_{s=s_0} \mathrm{Id}_p = 0$. Does this imply $\langle \gamma'(s_0), \sigma'_{s_0}(d(s_0)) \rangle = \left\langle \frac{\partial h}{\partial s}(s_0,d(s_0)), \frac{\partial h}{\partial t}(s_0,d(s_0)) \right\rangle = 0$ for any $s_0 \in \mathbb{R}$?

The resolution to this paradox relies on the observation of the distinction among three different types of variations. The first variation, denoted as $h_1(s,t)$, satisfying our desire that $h_1(s, d(s)) = \gamma(s)$, is actually what is provided in the original problem in the form $h_1(s,t) = \sigma_s(t)$. This variation is indeed defined on an irregular domain





because in this variation via geodesics we fix $\left|\frac{\partial h_1}{\partial t}(s,t)\right| \equiv 1$ for all (s,t) in the domain of definition, or equivalently speaking all geodesics travel with the same normalized speed, which means that they have to arrive at γ after different time periods since the distances $d(s)$ they travel through are generally not identical. This variation satisfies $h_1(s, d(s)) = \gamma(s)$, and hence $\frac{d}{ds}\big|_{s=s_0} h_1(s, d(s)) = \gamma'(s_0)$. However, in the paradox above we used $\frac{\partial h_1}{\partial s}(s_0, d(s_0))$, which equals $\frac{\partial}{\partial s}\big|_{s=s_0} h_1(s, d(s_0))$, is definitely different from $\gamma'(s_0) = \frac{d}{ds}\big|_{s=s_0} h_1(s, d(s))$, since one first fixes $t = d(s_0)$, forming the wavefront $s \mapsto h_1(s, d(s_0))$, and then consider the velocity of this wavefront curve. At moment $t = d(s_0)$, however, not every geodesic in the variation is lying on γ: some may have already passed through γ, some may still be on their way to γ. The wavefront is actually identical to part of the geodesic sphere, i.e. part of the boundary of the geodesic ball $B_{d(s_0)}(p)$. Recalling the Gauss Lemma in Chapter 3, whose proof we replicated in the paradox, it looks completely reasonable that $\frac{\partial h_1}{\partial s}(s_0, d(s_0))$ is orthogonal to $\sigma_{s_0}'(d(s_0))$. To see everything more clearly, let's compare $h_1(s,t)$ with the second variation $h_2(s,t) = \sigma_s(t)$, with the same formula of definition but a different domain of definition



In the overlapping area of this domain and the domain of h_1, the two variations h_1, h_2 must coincide. It is now clear that the wavefront of h_2 propagates with constant speed, thus maintaining the spherical shape of the wavefront. It is now not surprising that the tangent vector to $s \mapsto h_1(s, d(s_0)) = h_2(s, d(s_0))$ has nothing to do with $\gamma'(s)$: there is nothing about γ appearing in the construction of h_2 at all ($d(s_0)$ is just seen as an arbitrary real number: one can replace it with anything else)! All these happen because we artificially normalized each geodesic in the variation, thus losing the opportunity to encode the information of γ into the wavefronts being propagated. The third variation, which we used in solving this problem, given by $h_3(s,t) = \exp_p(tW(s))$, where $W(s)$ is a smooth curve in $T_p\tilde{M}$ satisfying $\exp_p(d(s_0)W(s)) = \gamma(s)$. Instead of staying normalized, $W(s)$ actually satisfies $|W(s)| = \frac{d(s)}{d(s_0)}$. Note that h_3 is defined on the same domain as that of h_2, but the wavefront of h_3 is different from that of h_2: at any time greater than zero, the wavefront of h_3 stays "in the shape of γ"! This probably explains why h_3 is used in this problem: it contains a lot of information about γ that we can utilize.

Remark 4: It is our strong feeling after solving this problem that life in a complete simply connected Riemannian manifold with non-positive sectional curvature is so similar to life in a totally normal neighborhood in many senses: one can have a variation of a geodesic via geodesics without worrying about conjugate points (see J. Milnor "Morse Theory" §14, P79-82 for descriptions of a variation via geodesics in a totally normal neighborhood); one can prove a "Gauss Lemma"; one can encode the change of distance between a fixed point and a point in motion along a curve into the change of the length of the speed vector of a geodesic joining those two points (like we did in the solution to this problem, or do Carmo did in proving the existence of totally normal neighborhoods in Chapter 3, §4, P74-77 (mainly in the proof of Lemma 4.1 on P75)); etc. Note that in Remark 3 we actually proved the Gauss Lemma for complete simply connected Riemannian manifold with non-positive sectional curvature, or equivalently speaking, the Gauss Lemma (Chapter 3, Lemma 3.5, on P69) applies to any complete simply connected Riemannian manifolds with non-positive sectional curvature.

Remark 5: Looking back into the variation we constructed in the solution to Exercise 1 of this chapter, it is understandable that we neglected some detail and thus have to work that out again.

Remark 6: At the end of this chapter, let us summarize some routine ways to construct a variation we have encountered so far. ① Find a vector field along the curve being considered, and use Proposition 2.2 in Chapter 9 (P193). For the sake of simplifying the formula for the second variation, it is prefered to construct a parallel vector field; if the curve under consideration is closed, one needs to take into consideration whether or

No.

Date

not this vector field is well-defined (say, in the situation where this vector field is constructed from parallel transporting a specified tangent vector in the tangent space to a fixed point in the curve, cf. Exercise 4 in this chapter). This is a typical "from variational field to variation" approach. ② Use the completeness of the manifold to form the variation directly by joining points with geodesics. If succeeds, one automatically obtain a variation via geodesics, whose variational field is necessarily a Jacobi field (cf. Exercise 2 in Chapter 5 on P119) which are easier to characterise (as seen in Solution to (a)(ii) of this Exercise in the step showing strict positiveness of the second variation of the energy). To successfully carry out this method one needs to take into consideration the potential existence of conjugate points along some certain geodesic (as the case in Exercise 1 of this chapter). Some curvature conditions may help to exclude the existence of conjugate points (like in this Exercise, non-positive sectional curvature brings in the Cartan-Hadamard Theorem into the play). This is a typical "from variation to variational field" approach.
③ Sometimes, in order for the variation to satisfy some special requirements, one needs to combine ① and ② organically (as seen in the proof for the Weinstein-Synge Theorem (Theorem 3.7 on P204-205), one adapts the technique for constructing a variational field along a closed geodesic (cf. Exercise 4 of this chapter) to construct a variational field which fits nicely between two boundary geodesics, which is essentially the first approach in this remark but also employs some careful observations on the behavior of geodesics as suggested in the second approach in this remark), or incorporate one approach into the other carefully with subtlety.