

Manifold Learning on Fibre Bundles: A Geometric Framework for Graph Data with Non-scalar Interactions

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Outline

Background

- ▶ Spectral Geometry and Data Analysis

Manifold Learning on Fibre Bundles

- ▶ Motivation: Comparative Biology
- ▶ A Fibre Bundle Vision
- ▶ Horizontal Diffusion Maps (HDM)

Other Projects

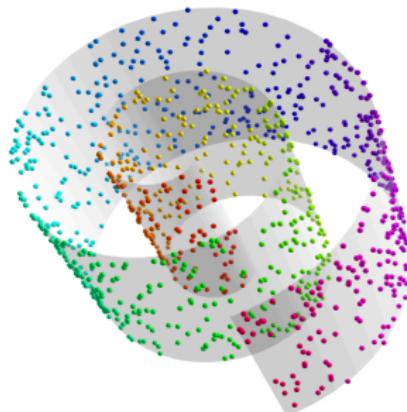
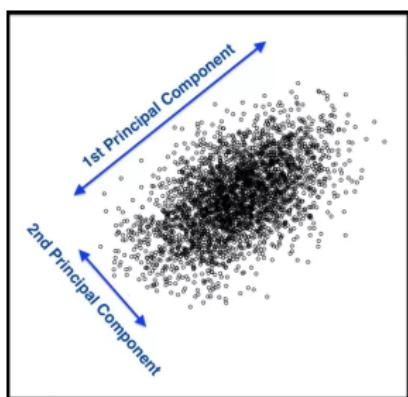
Manifold Learning: Discovering *Intrinsic* Structures

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Manifold Learning: *Nonlinear* Dimension Reduction

- ▶ **Curse of Dimensionality:** For D -dimensional data, need $\Omega(\epsilon^{-1} \exp(D \log \frac{1}{\epsilon}))$ samples to ensure estimation error $\leq \epsilon$
- ▶ **Manifold Assumption:** Data lie approximately on a d -dimensional submanifold of \mathbb{R}^D , with $d \ll D$; expecting sampling complexity $\Omega(\epsilon^{-1} \exp(d \log \frac{1}{\epsilon}))$ instead



Manifold Learning and Discrete Geometry – Various Interpretations

- ▶ **Nonparametric Statistics:** Regression and density estimation under manifold assumptions (L. Wasserman, P. Bickel et al.)
- ▶ **Gaussian Processes:** Gaussian process latent variable models (N. Lawrence et al.)
- ▶ **Coarse Geometry:** Geometric Whitney Problem – Metric space approximation under the Gromov–Hausdorff distance (C. Fefferman et al.)
- ▶ **Finite Elements:** Discrete exterior calculus (P. Schröder, M. Desbrun, D. Arnold et al.)

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- ▶ **Spectral Geometry:** Laplacian Eigenmaps (M. Belkin & P. Niyogi, S. Lafon & R. Coifman, M. Maggioni et al.)

Spectral Geometry

CAN ONE HEAR THE SHAPE OF A DRUM?

MARK KAC, The Rockefeller University, New York

To George Eugene Uhlenbeck on the occasion of his sixty-fifth birthday

“La Physique ne nous donne pas seulement l’occasion de résoudre des problèmes . . . , elle nous fait présentir la solution.” H. POINCARÉ.

- Mark Kac. “Can One Hear the Shape of a Drum?” *The American Mathematical Monthly*, 73, no. 4P2 (1966): 1–23.

Spectral Geometry: Determining the Metric Tensor on M

$$\Delta_M u_n = -\lambda_n u_n, \quad n = 0, 1, 2, \dots$$

$$0 = \lambda_0 < \lambda_1 \leq \lambda_2 \leq \dots \nearrow \infty$$

- ▶ Does knowing all λ_n 's determine M up to isometry?
(Spoiler Alert: No)

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- ▶ Does knowing all λ_n 's determine M up to isometry?
(Spoiler Alert: No)
- ▶ Isospectral but non-isometric:
 - ▶ Flat tori (Milnor, 1964)
 - ▶ Riemann surfaces with const. neg. curvature (Vignéras, 1980)
 - ▶ Lens spaces with const. curvature (Ikeda, 1983)
 - ▶ Riemannian covering spaces (Sunada, 1985)
 - ▶ Counterexamples in 2D (Gordon–Webb–Wolpert, 1992)
- ▶ Cospectral graphs

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- ▶ Does knowing all λ_n 's **and all** $\{u_n\}$'s determine M up to isometry?

Yes – via the *heat kernel*: $\forall x, y \in M, t \in [0, \infty)$

$$k_M(x, y; t) = \sum_{n=0}^{\infty} e^{-\lambda_n t} u_n(x) u_n(y)$$

- ▶ Heat kernel completely determines the metric up to isometry
[Varadhan's Formula:
 $\lim_{t \rightarrow 0+} 4t \log k_M(x, y; t) = -d_M^2(x, y)]$

Spectral Embedding

(Bérard–Besson–Gallot, 1994) Any closed Riemannian manifold in

$$\mathcal{M}_{n,k,D} = \left\{ (M, g) \mid \dim(M) = n, \text{Ric}(g) \geq (n-1)kg, \text{diam}(M) \leq D \right\}$$

can be embedded into the infinite dimensional Hilbert space ℓ^2 using the *heat kernel map*

$$M \ni x \longmapsto \Phi_t(x) := \left(e^{-\lambda_0 t/2} u_0(x), e^{-\lambda_1 t/2} u_1(x), \dots \right) \in \ell^2$$

- ▶ $\langle \Phi_t(x), \Phi_t(y) \rangle_{\ell^2} = k_M(x, y; t)$ RKHS
- ▶ Estimates for $k_M \leftrightarrow$ (Gromov) Precompactness of $\mathcal{M}_{n,k,D}$

Spectral Embedding: Beyond BBG'94

- ▶ Improving the BBG'94 spectral embedding:
 - ▶ (**Jones–Maggioni–Schul, 2010**) Local, **finitely** many eigenfunctions used to make **bi-Lipschitz** coordinate charts
 - ▶ (**Bates, 2014**) Make (Jones–Maggioni–Schul, 2010) **global**
 - ▶ (**Wang–Zhu, 2015**) Heat kernel maps can be made **isometric** using Nasher–Moser
 - ▶ (**Portegies, 2016**) **Global**, **finite**, and **almost isometric** embedding using harmonic radius arguments
- ▶ Embedding tangent bundles using the connection Laplacian:
 - ▶ (**Wu, 2017**) BGG'94-type embedding, but with the heat kernels of the **connection Laplacian** (rough Laplacian)
 - ▶ (**Lin–Wu, 2018**) Make (Wu, 2017) **finite**

Geometric Data Analysis

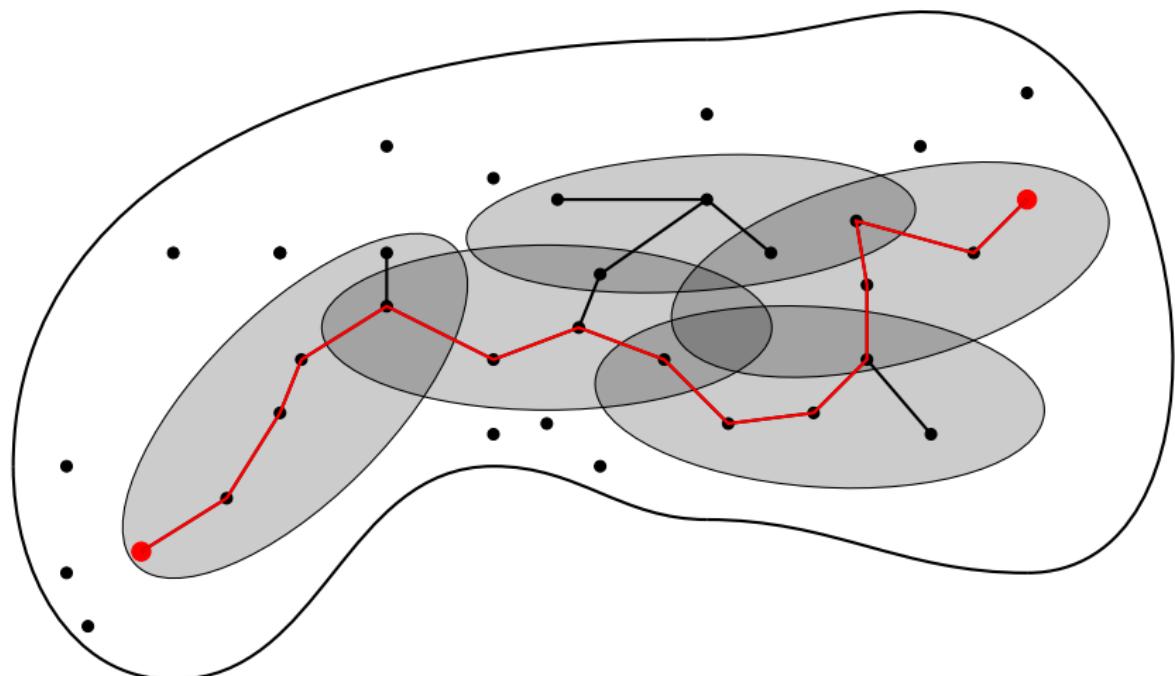
Underpinning methodology (**Lim, 2019**):

Graphs are discrete Riemannian manifolds

Manifold	Graph
tangent vectors	edges
Laplacian	graph Laplacian
heat kernel	heat kernel
diffusion process	random walk

- Lek-Heng Lim. "Hodge Laplacians on Graphs." *SIAM Review*, to appear (2019)
- Mikhail Belkin, Partha Niyogi. "Laplacian Eigenmaps for Dimensionality Reduction and Data Representation." *Neural Computation*, 15 (6), 1373–1396 (2003)
- Ronald Coifman, Stéphane Lafon. "Diffusion Maps." *Applied and Computational Harmonic Analysis*, 21 (1), 5–30 (2006)

Random Walks “Knit Together” Local Geometry



Small distances are much more reliable!

Diffusion Maps

- ▶ Data $\mathcal{X} := \{x_1, \dots, x_n\} \subset \mathbb{R}^D$, point cloud

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$$W_{ij} = \exp\left(-\|x_i - x_j\|^2 / \epsilon\right)$$

and diagonal matrix $D \in \mathbb{R}^{n \times n}$ with

$$D_{ii} = \sum_{j=1}^n W_{ij}$$

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- ▶ Build **graph random walk Laplacian** $L = I - D^{-1}W$, and perform eigen-decomposition

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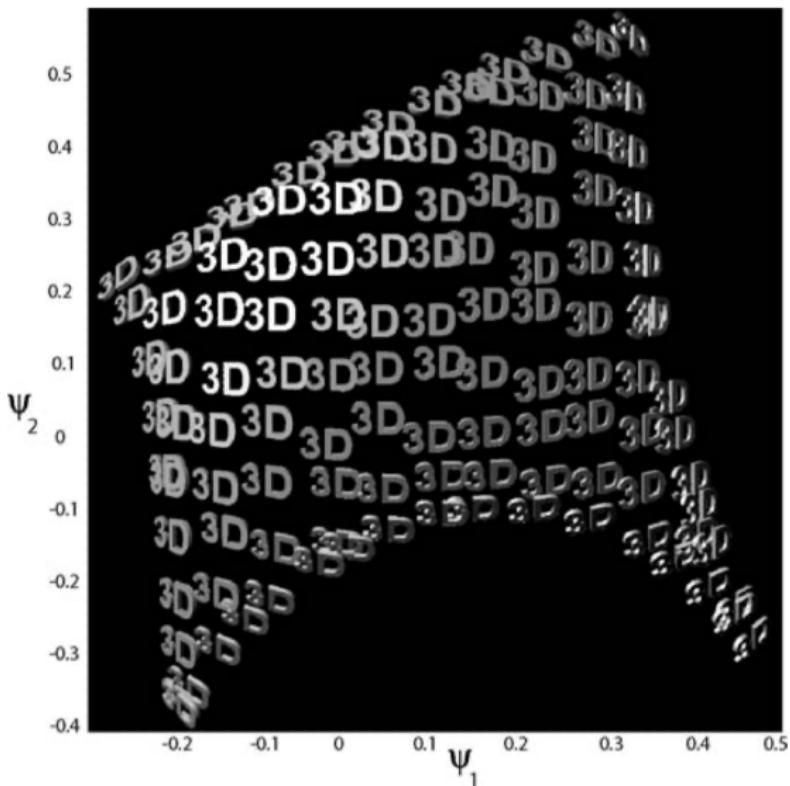
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- ▶ Build **graph random walk Laplacian** $L = I - D^{-1}W$, and perform eigen-decomposition

$$Lu_i = \lambda_i u_i, \quad i = 1, \dots, n$$

- ▶ For any $1 \leq d \leq n$, embed \mathcal{X} into \mathbb{R}^d by

$$x_k \mapsto \left(\lambda_1^{1/2} u_1(k), \dots, \lambda_d^{1/2} u_d(k)\right)$$



Continuous Limit

- ▶ Explicitly, $D^{-1}W$ is a Markov operator: for any $v \in \mathbb{R}^n$,

$$(D^{-1}Wv)_i = \frac{\sum_{j=1}^n \exp\left(-\|x_i - x_j\|^2/\epsilon\right) v_j}{\sum_{j=1}^n \exp\left(-\|x_i - x_j\|^2/\epsilon\right)}$$

- ▶ As $n \rightarrow \infty$, $D^{-1}W$ converges weakly to the integral operator

$$P_\epsilon f(x) = \frac{\int_M \exp\left(-\|x - y\|^2/\epsilon\right) f(y) \, d\text{vol}(y)}{\int_M \exp\left(-\|x - y\|^2/\epsilon\right) \, d\text{vol}(y)}$$

for all $f \in L^1(M)$.

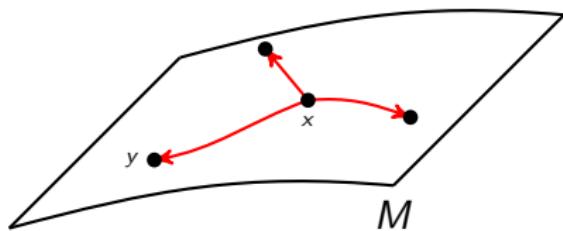
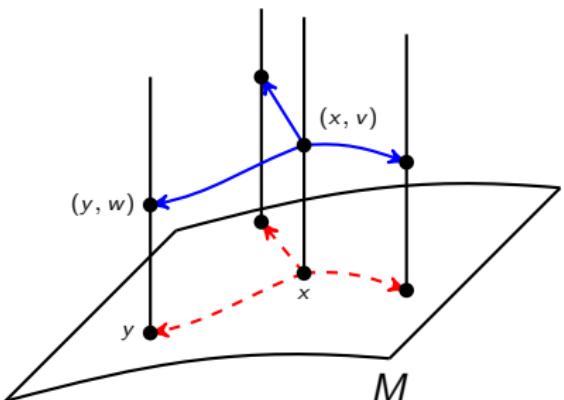
Asymptotic Theory, Convergence Rates, etc.

Theorem (Belkin–Niyogi, 2005; Coifman–Lafon, 2006). For $f \in C^\infty(M)$ and $x \in M$, if $\{x_i\}_{i=1}^n \stackrel{\text{i.i.d.}}{\sim} p \, d\text{vol}_M$, the infinitesimal generator of the limiting diffusion process is $p \cdot \Delta_M + 2\nabla p \cdot \nabla$.

Theorem (Singer, 2006). If $\{x_i\}_{i=1}^n$ i.i.d. uniform on manifold M of dim. d , $D^{-1}W$ converges to P_ϵ at rate $O(n^{-1/2}\epsilon^{1/2-d/4})$ in the weak sense. For non-uniform samples the rate is $O(n^{-1/2}\epsilon^{-d/4})$.

Theorem (N. García Trillo et. al., 2018+). Eigenvalues and eigenvectors of the graph Laplacian converge to their counterparts of the Laplace–Beltrami operator at rate $O(n^{-1/(d+4)})$.

This Talk: *Horizontal* Diffusion Maps on Fibre Bundles

 P_ϵ  $H_{\epsilon, \delta}$ 

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Other Projects

Morphometrics



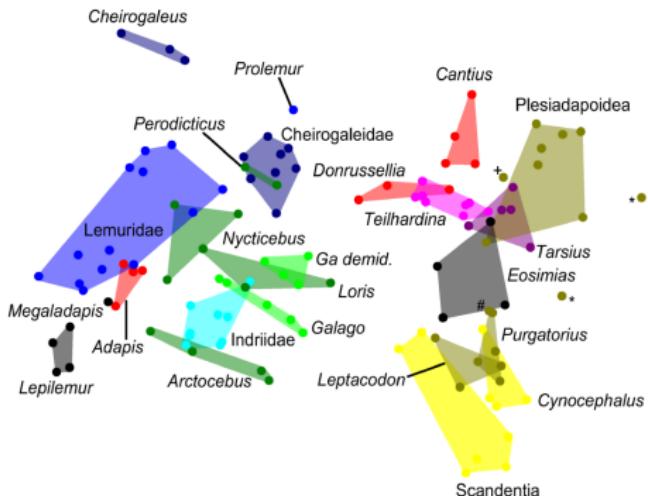
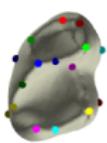
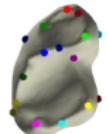
<i>Iris setosa</i>				<i>Iris versicolor</i>				<i>Iris virginica</i>			
Sepal length	Sepal width	Petal length	Petal width	Sepal length	Sepal width	Petal length	Petal width	Sepal length	Sepal width	Petal length	Petal width
5·1	3·5	1·4	0·2	7·0	3·2	4·7	1·4	6·3	3·3	6·0	2·5
4·9	3·0	1·4	0·2	6·4	3·2	4·5	1·5	5·8	2·7	5·1	1·9
4·7	3·2	1·3	0·2	6·9	3·1	4·9	1·5	7·1	3·0	5·9	2·1
4·6	3·1	1·5	0·2	5·5	2·3	4·0	1·3	6·3	2·9	5·6	1·8
5·0	3·6	1·4	0·2	6·5	2·8	4·6	1·5	6·5	3·0	5·8	2·2
5·4	3·9	1·7	0·4	5·7	2·8	4·5	1·3	7·6	3·0	6·6	2·1
4·6	3·4	1·4	0·3	6·3	3·3	4·7	1·6	4·9	2·5	4·5	1·7
5·0	3·4	1·5	0·2	4·9	2·4	3·3	1·0	7·3	2·9	6·3	1·8

- Ronald A. Fisher. "The Use of Multiple Measurements in Taxonomic Problems." *Annals of Eugenics*, 7.2, 179–188 (1936)

Landmark-Based Geometric Morphometrics

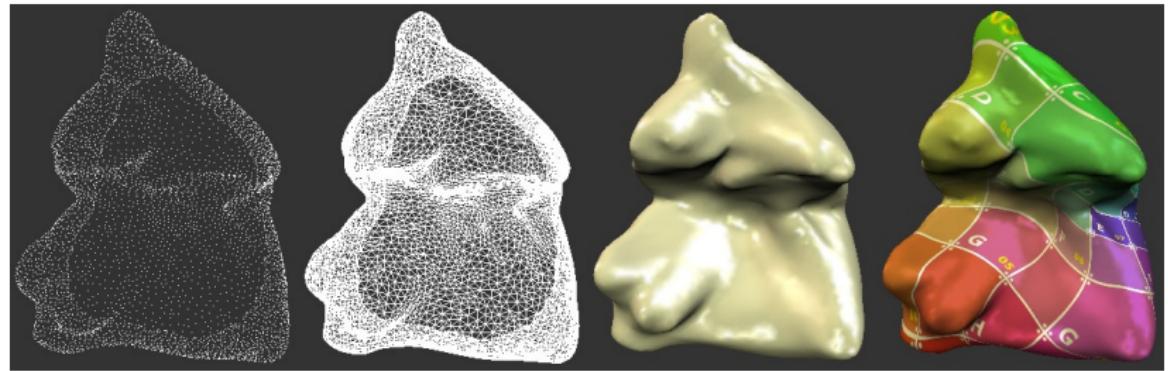
$$\left(S_1, \{x_j\}_{j=1}^J \right), \left(S_2, \{y_j\}_{j=1}^J \right) \rightarrow$$

$$d_{Procrustes}^2(S_1, S_2) = \min_{R \text{ rigid motion}} \frac{1}{J} \sum_{j=1}^J \|R(x_j) - y_j\|^2$$



Boyer et al. "Algorithms to Automatically Quantify the Geometric Similarity of Anatomical Surfaces." *Proceedings of the National Academy of Sciences*, 108.45 (2011): 18221-18226.

Data Acquisition: microCT (High-Resolution X-ray CT)



Surface reconstructed from μ CT-scanned voxel data



Sharing the Bones

Duke researchers bring digital tools to the Stone Age findings of the Rising Star cave expedition

Writer: Louise Flynn
December 11, 2015



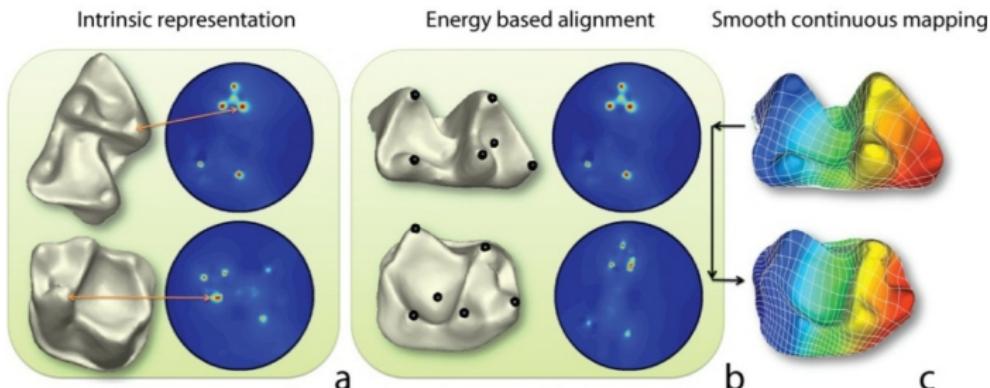
Streaked with ancient limestone dust, the scientist hovering near Steve Churchill's worktable waited to see his reaction to the fossil she'd surgically removed from its eons-old resting place eighty meters below ground. Crammed in the tent's open window, a camera crew from *National Geographic* gathered more footage. Researchers came in and out,



“Big Data” for Biologists: Impossible to landmark them all!!

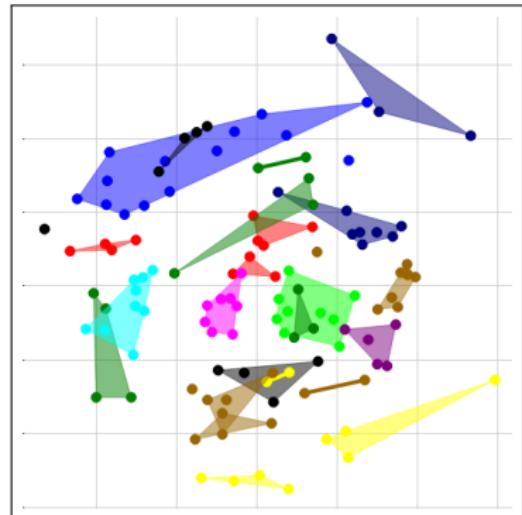
Landmarks \leftrightarrow Diffeomorphisms

$$d_{\text{CP}}^2(S_i, S_j) = \inf_{\mathcal{C} \in \mathcal{A}(S_i, S_j)} \inf_{R \in \mathbb{E}(3)} \int_{S_i} \|R(x) - \mathcal{C}(x)\|^2 d\text{vol}_{S_i}(x)$$

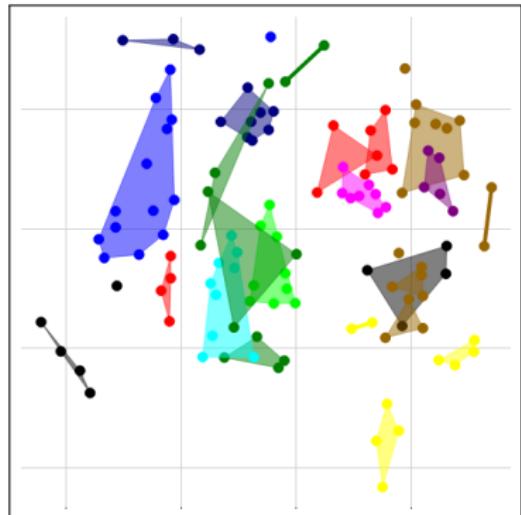


- Reema Al-Aifari, Ingrid Daubechies, and Yaron Lipman. "Continuous Procrustes Distance Between Two Surfaces." *Communications on Pure and Applied Mathematics*, 66, no. 6 (2013): 934–964.

Multi-Dimensional Scaling (MDS)



landmark-free



landmark

Outline

Background

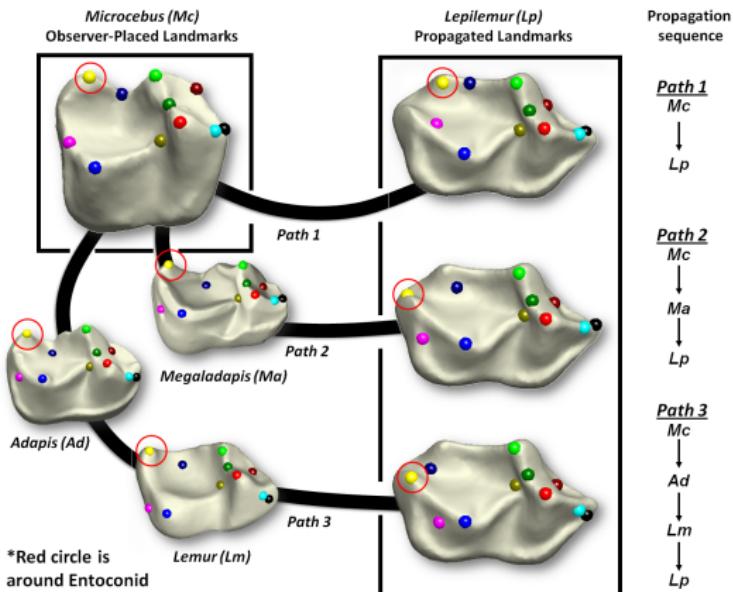
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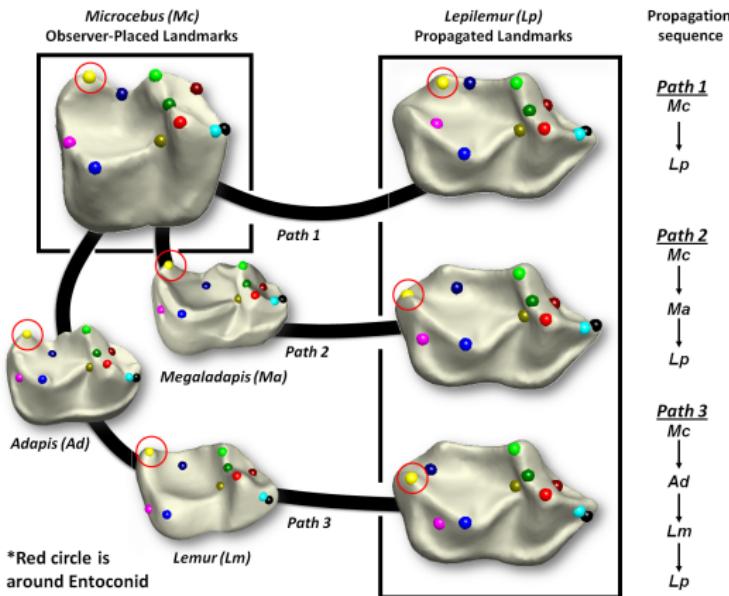
Interpretability Issue



Automate everything biologists do, including making mistakes...

- Boyer et al. "Algorithms to Automatically Quantify the Geometric Similarity of Anatomical Surface." *Proceedings of the National Academy of Sciences*, 108.45 (2011): 18221–18226

Trust Small Distances — Let Diffusion Maps Do it for You!

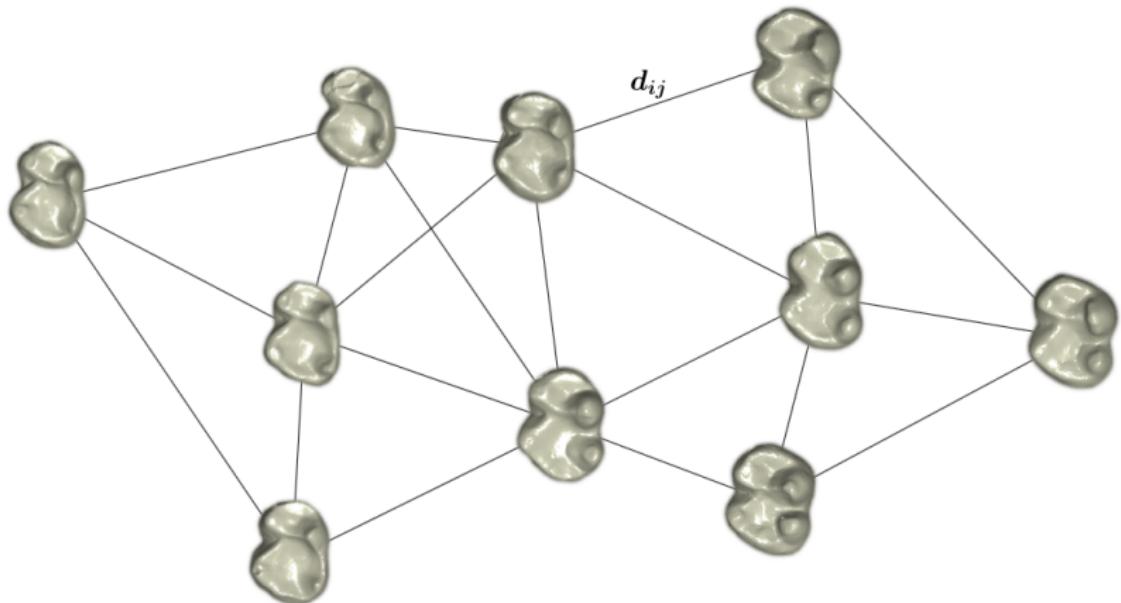


"Correct" like a
biologist, but
automatically?

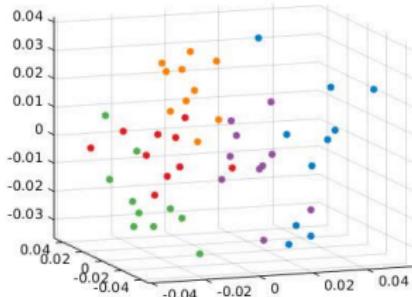
small distances between $S_1, S_2 \rightarrow$ OK maps
larger distances \rightarrow not OK

- **Tingran Gao**, Gabriel S. Yapuncich, Ingrid Daubechies, Sayan Mukherjee, Doug M. Boyer. "Development and Assessment of Fully Automated and Globally Transitive Geometric Morphometric Methods, with Application to a Biological Comparative Dataset with High Interspecific Variation." *The Anatomical Record: Advances in Integrative Anatomy and Evolutionary Biology*, 301 (4), 636–658 (2018)

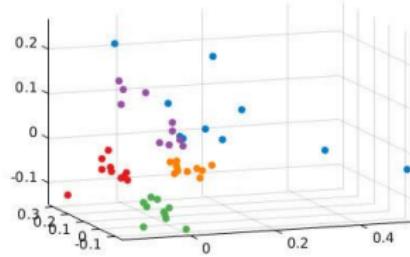
Trust Only *Small* Distances: Geodesics in Shape Space



MDS: 50 Molars of Madagascar Lemurs from 5 Groups



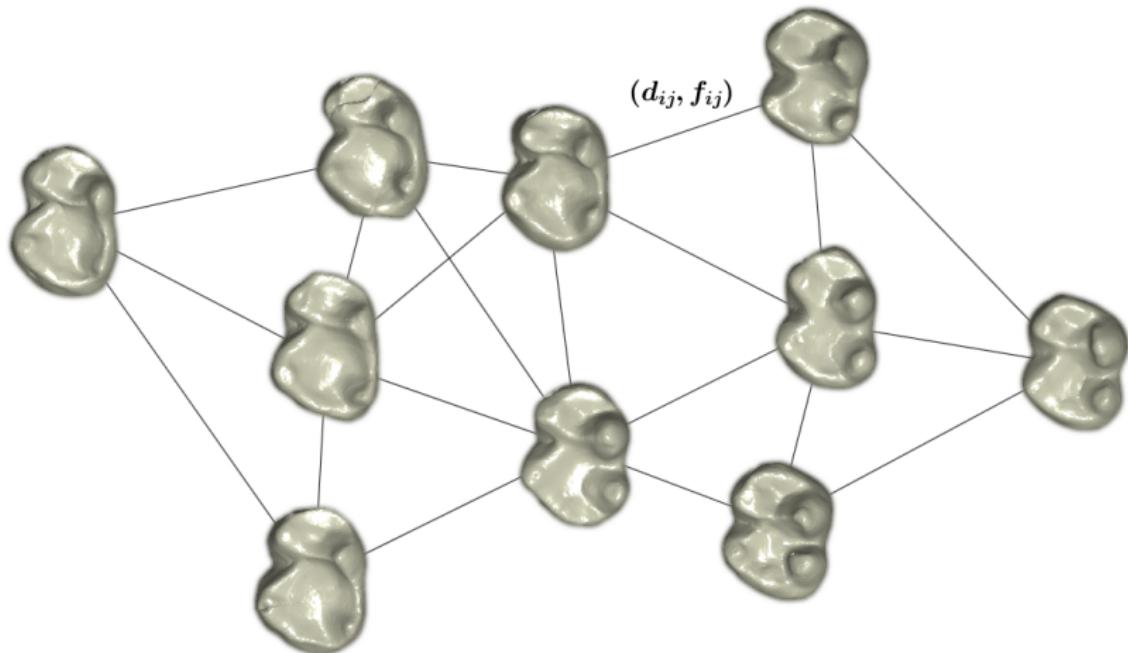
Continuous Procrustes Distance



Diffusion Distance

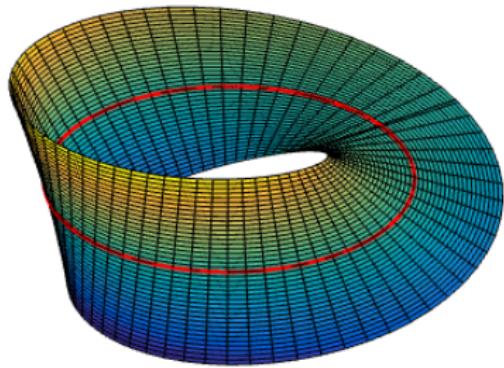
- Still, we haven't used the correspondence maps!

Geometric Model — *Parallel Transport* on Fibre Bundles

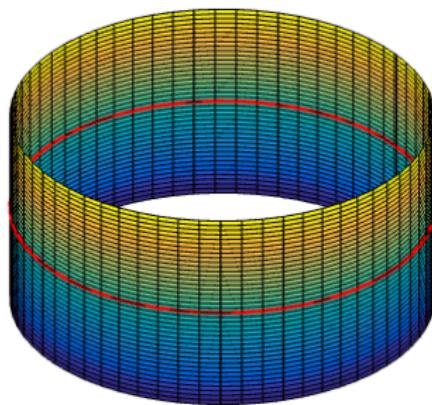


Fibre Bundle $\mathcal{E} = (E, M, F, \pi)$

- ▶ M : base manifold
- ▶ F : fibre manifold
- ▶ E : total manifold
- ▶ $\pi : E \rightarrow M$: smooth surjective map (*bundle projection*)
- ▶ *local triviality*: for “small” open set $U \subset M$, $\pi^{-1}(U)$ is diffeomorphic to $U \times F$

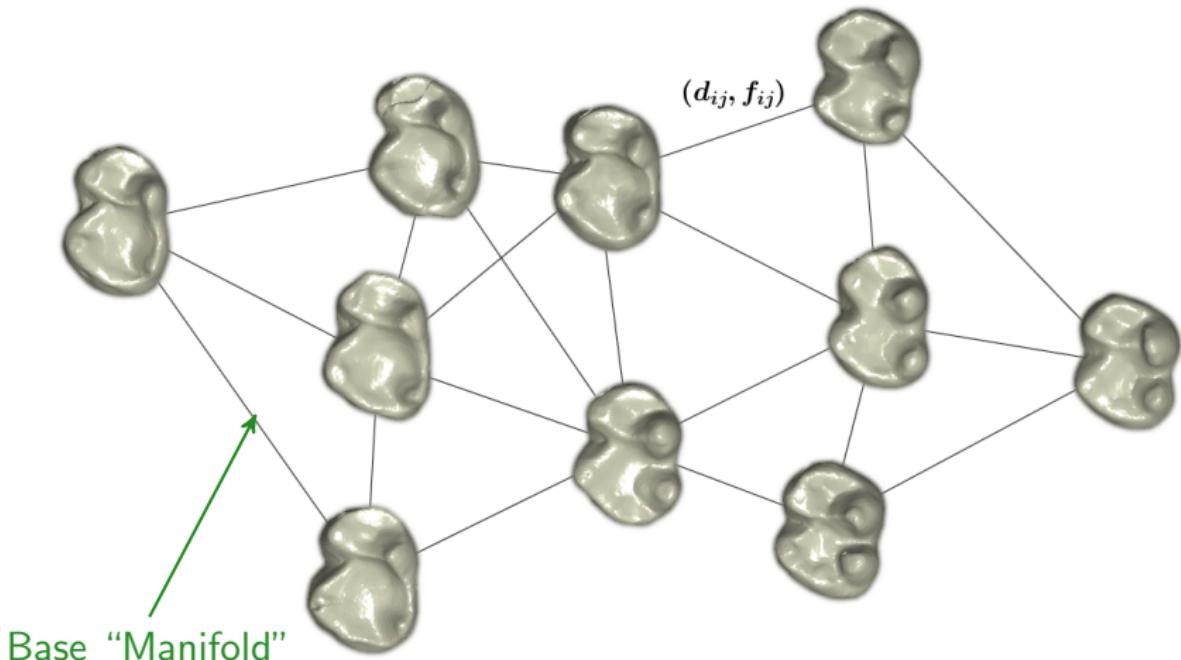


A non-trivial fibre bundle

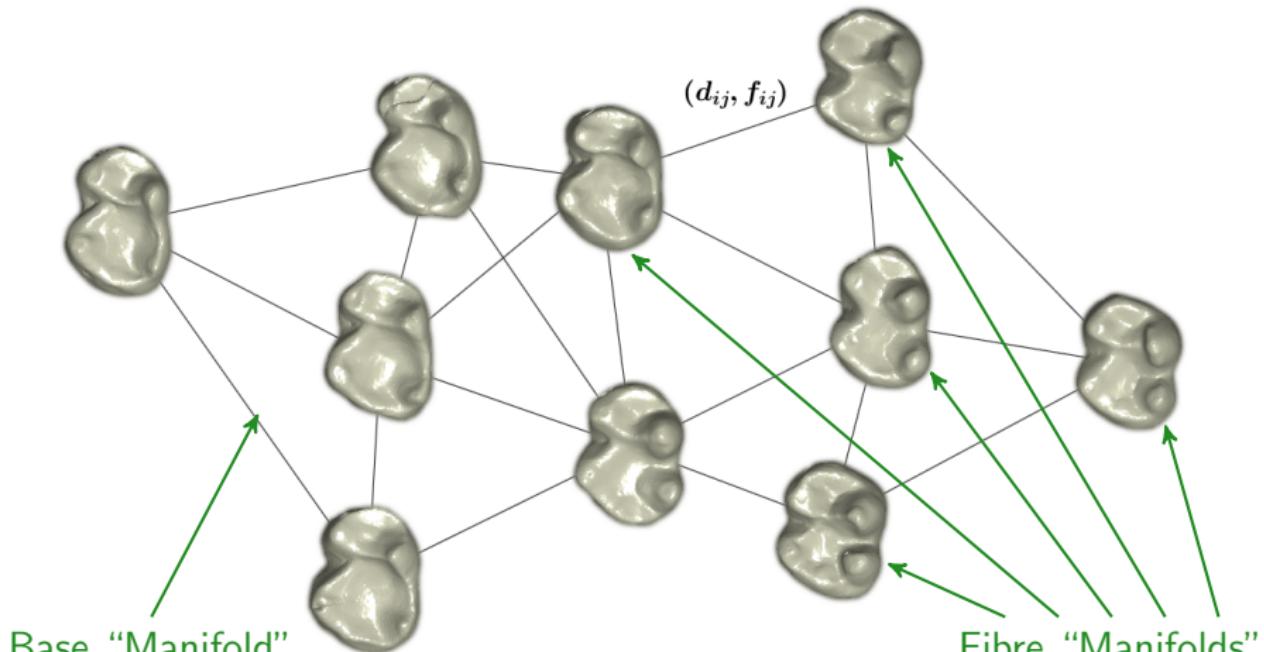


A trivial fibre bundle

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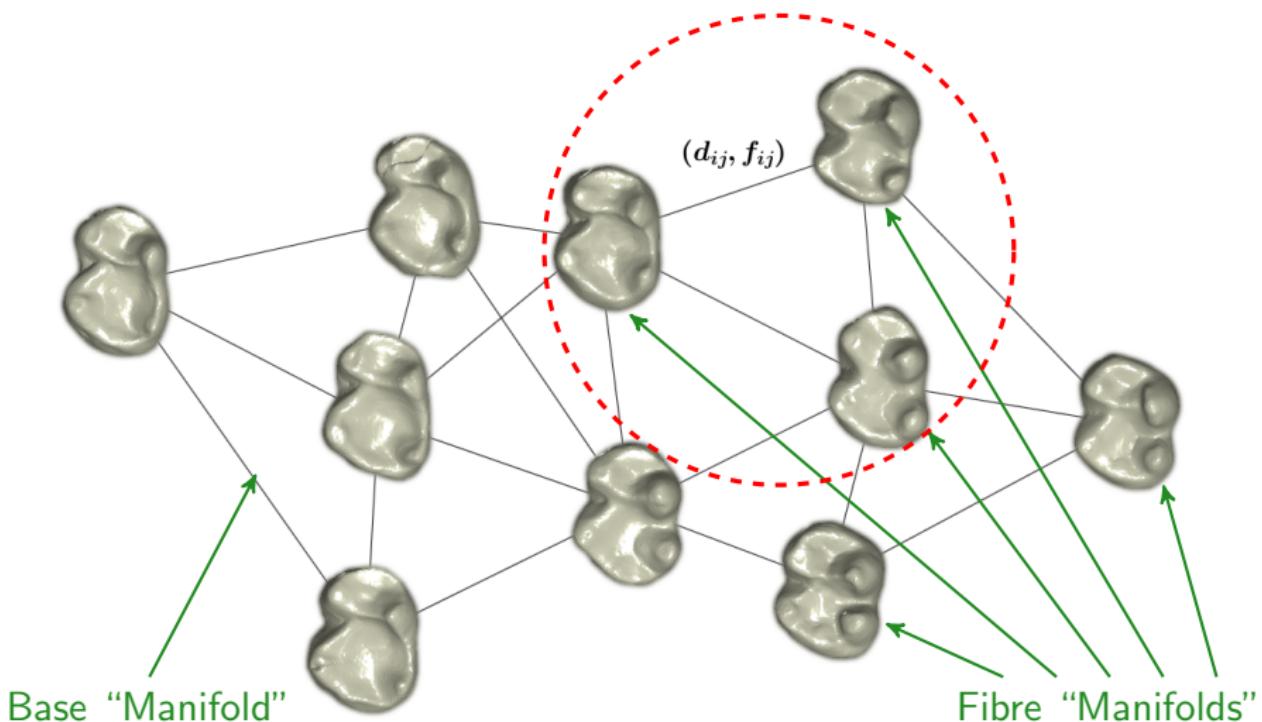


Geometric Model — *Parallel Transport* on Fibre Bundles

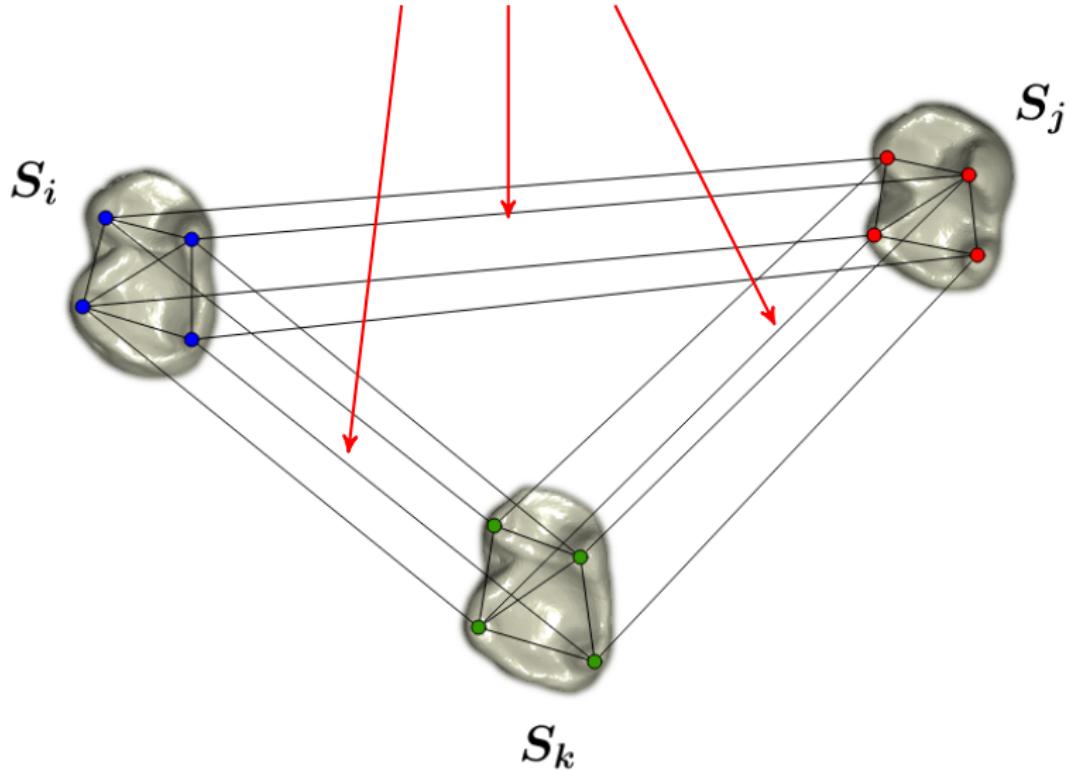


This is a (disk-type) surface bundle!

Geometric Model — *Parallel Transport* on Fibre Bundles

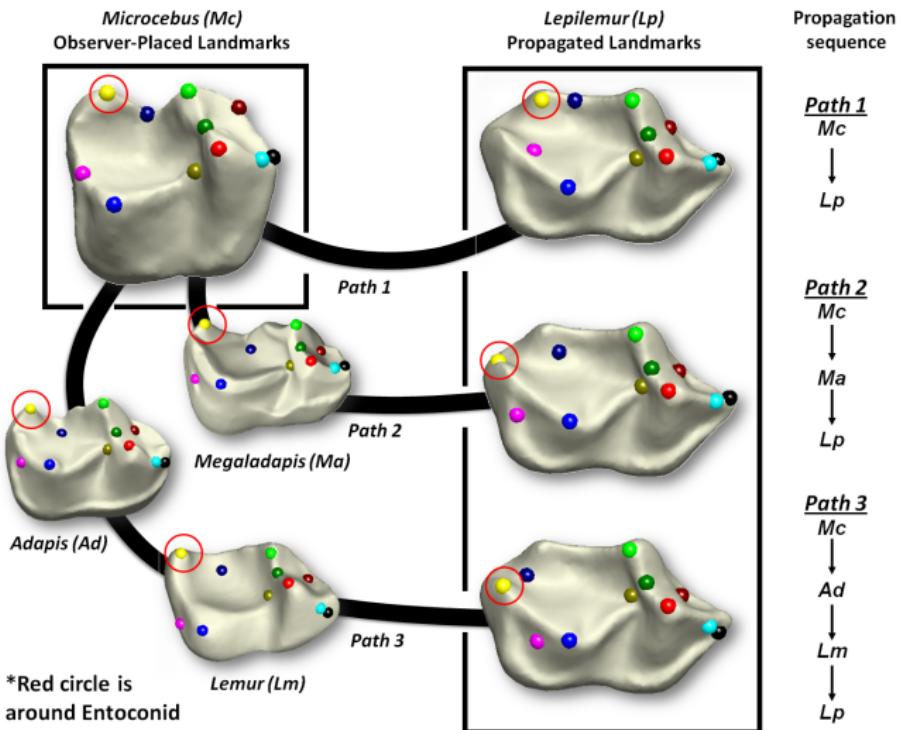


Geometric Model — *Parallel Transport* on Fibre Bundles



Correspondence maps viewed as parallel-transports between fibres!

Shape Space is NOT a Trivial Fibre Bundle



Outline

Background

- ▶ Spectral Geometry and Data Analysis

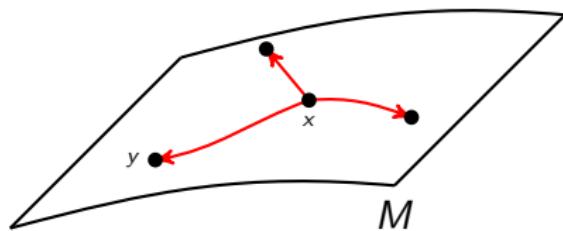
Manifold Learning on Fibre Bundles

- ▶ Motivation: Comparative Biology
- ▶ A Fibre Bundle Vision
- ▶ **Horizontal Diffusion Maps (HDM)**

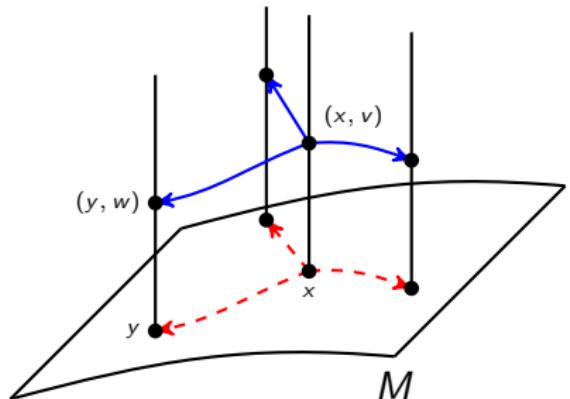
Other Projects

Parallel Transport Horizontally Lifts a Random Walk

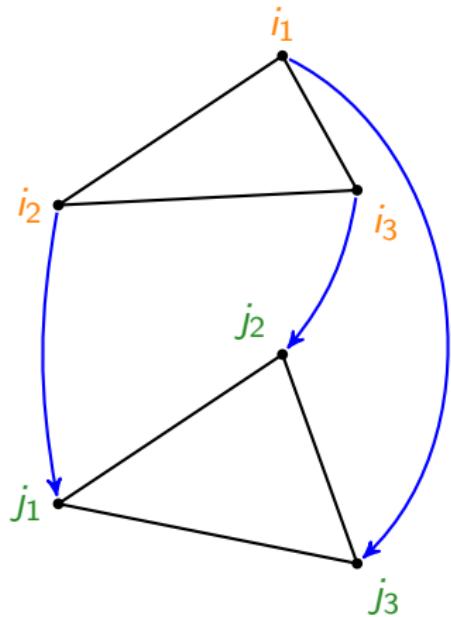
P_ϵ



$H_{\epsilon,\delta}$

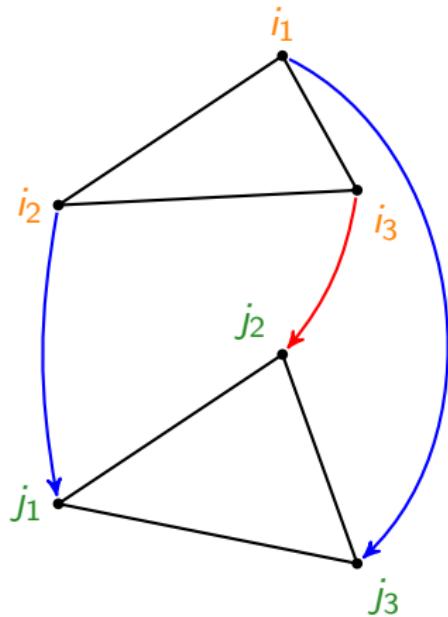


Map = Parallel Transport + Vertical Diffusion



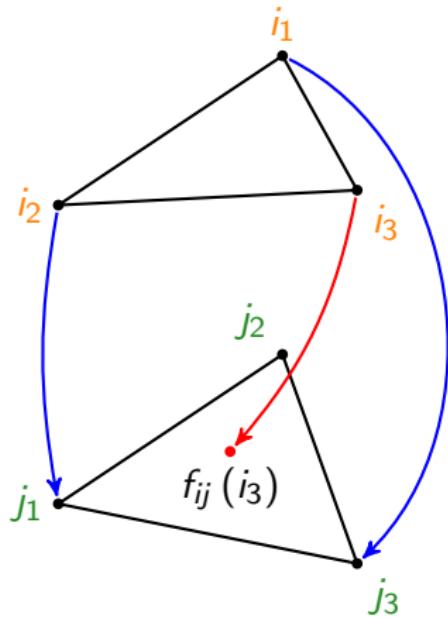
$$S_j \xrightarrow{\quad} \cdots \ j_1 \ j_2 \ j_3 \ \cdots$$
$$S_i \downarrow \begin{pmatrix} \vdots & \vdots & \vdots & \vdots \\ i_1 & \cdots & 0 & 0 & 1 & \cdots \\ i_2 & \cdots & 1 & 0 & 0 & \cdots \\ i_3 & \cdots & 0 & 1 & 0 & \cdots \\ \vdots & \vdots & \vdots & \vdots & \vdots \end{pmatrix}$$

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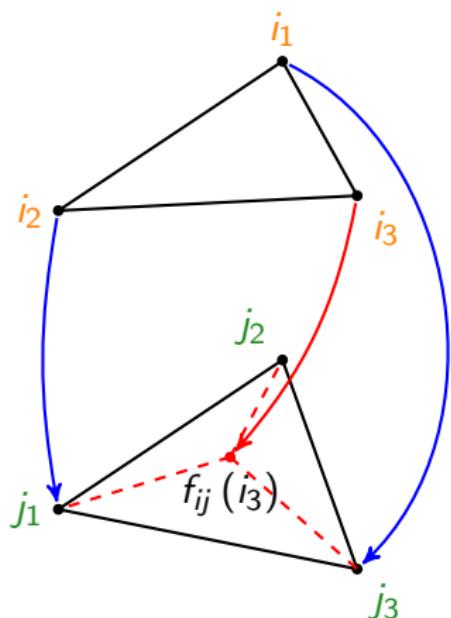
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Map = Parallel Transport + Vertical Diffusion



$$S_i \downarrow \begin{array}{cccc} & S_j & & \\ \cdots & j_1 & j_2 & j_3 & \cdots \\ \vdots & \vdots & \vdots & \vdots & \\ i_1 & 0 & 0 & 1 & \cdots \\ i_2 & 1 & 0 & 0 & \cdots \\ i_3 & 0.91 & 0.95 & 0.88 & \cdots \\ \vdots & \vdots & \vdots & \vdots & \end{array}$$

$$\rho_{ij}^{\delta}(r,s) = \exp\left(-\frac{\|f_{ij}(i_r) - j_s\|^2}{\delta}\right)$$

Horizontal Diffusion Maps (HDM) – Matrix Form

Assume there are n shapes, S_1, \dots, S_n , and each S_i has ℓ_i vertices.

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- ▶ Let \mathcal{W} be an n -by- n **block** matrix, with (i, j) th block

$$\exp\left(-\frac{d_{\text{cP}}^2(S_i, S_j)}{\epsilon}\right) \cdot \rho_{ij}^{\delta} \in \mathbb{R}^{\ell_j \times \ell_i}$$

ϵ : horizontal bandwidth parameter

δ : vertical bandwidth parameter

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- ▶ Let \mathcal{D} be a diagonal matrix, with k th diagonal entry equaling the k th row sum of \mathcal{W}
- ▶ **Horizontal Random Walk Laplacian:** $I - \mathcal{D}^{-1}\mathcal{W}$

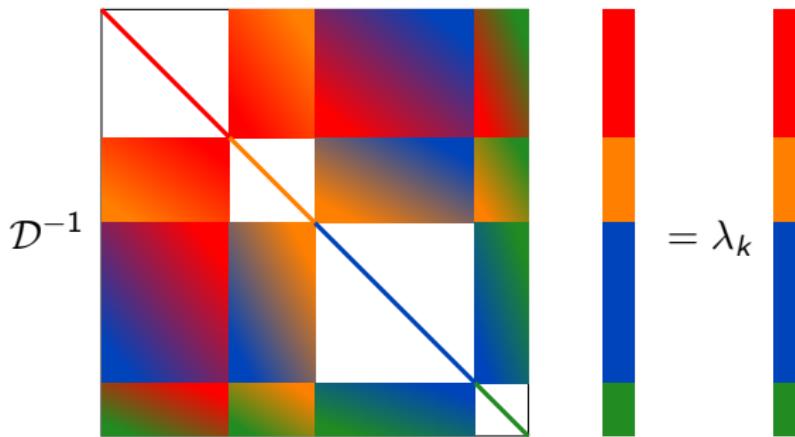
Remark. \mathcal{W} can be viewed as a flattening of a 4-tensor

Horizontal Diffusion Maps (HDM) – Matrix Form

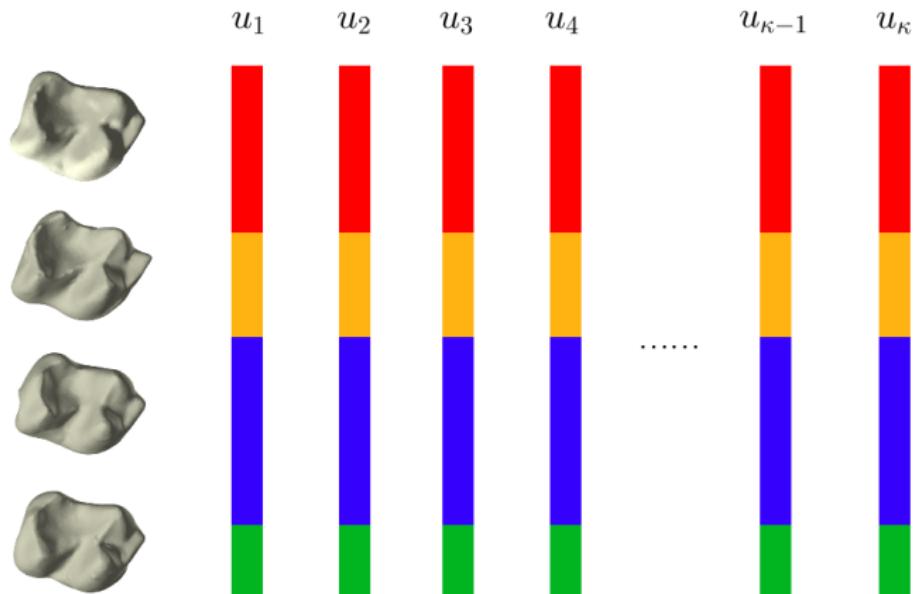
For all $1 \leq k \leq \kappa$, solve eigenproblems:

$$\mathcal{D}^{-1}\mathcal{W}$$

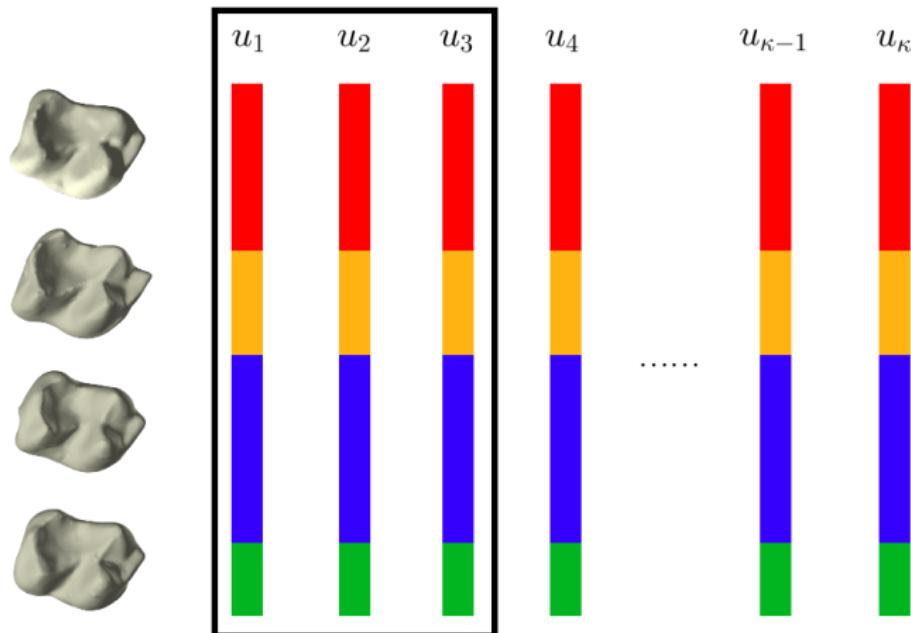
$$u_k = \lambda_k u_k,$$



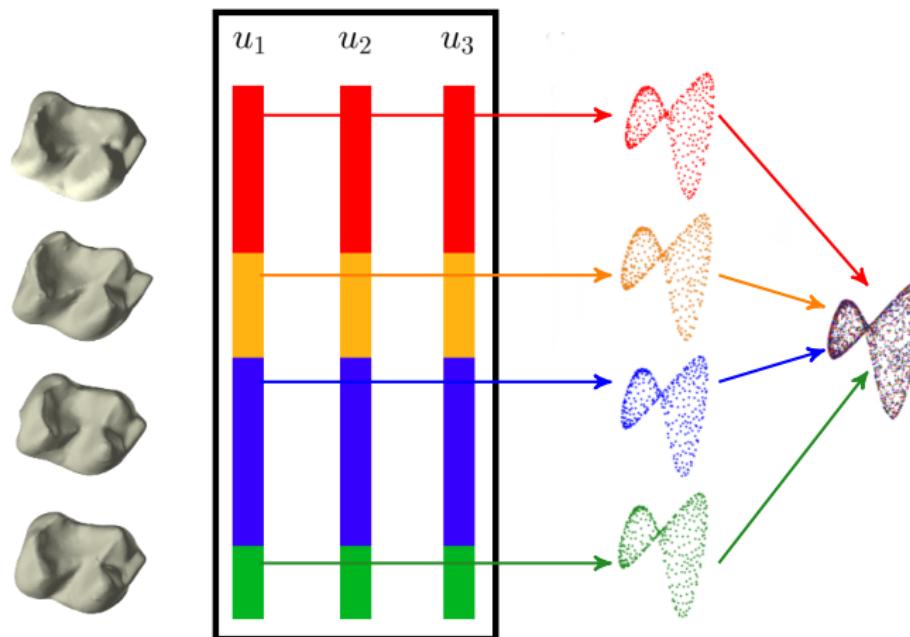
HDM Task 1: Embedding the Entire Bundle



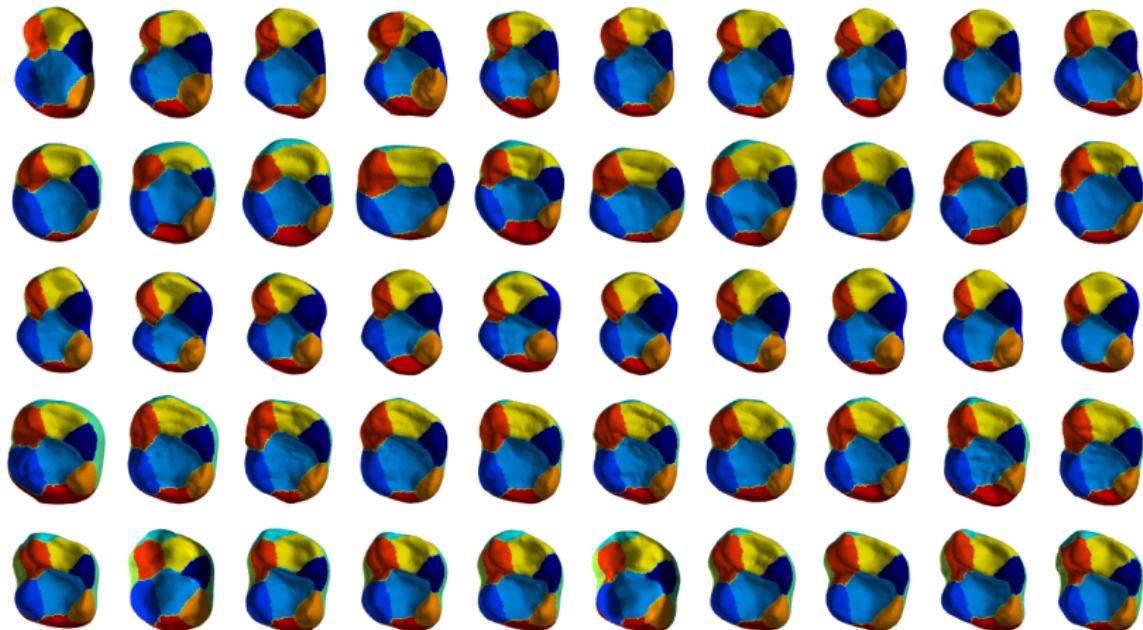
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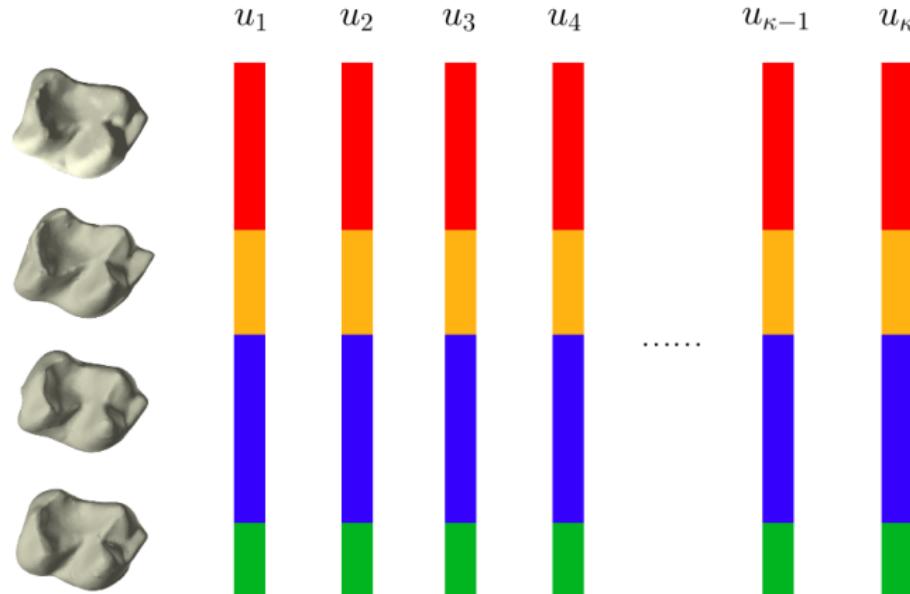
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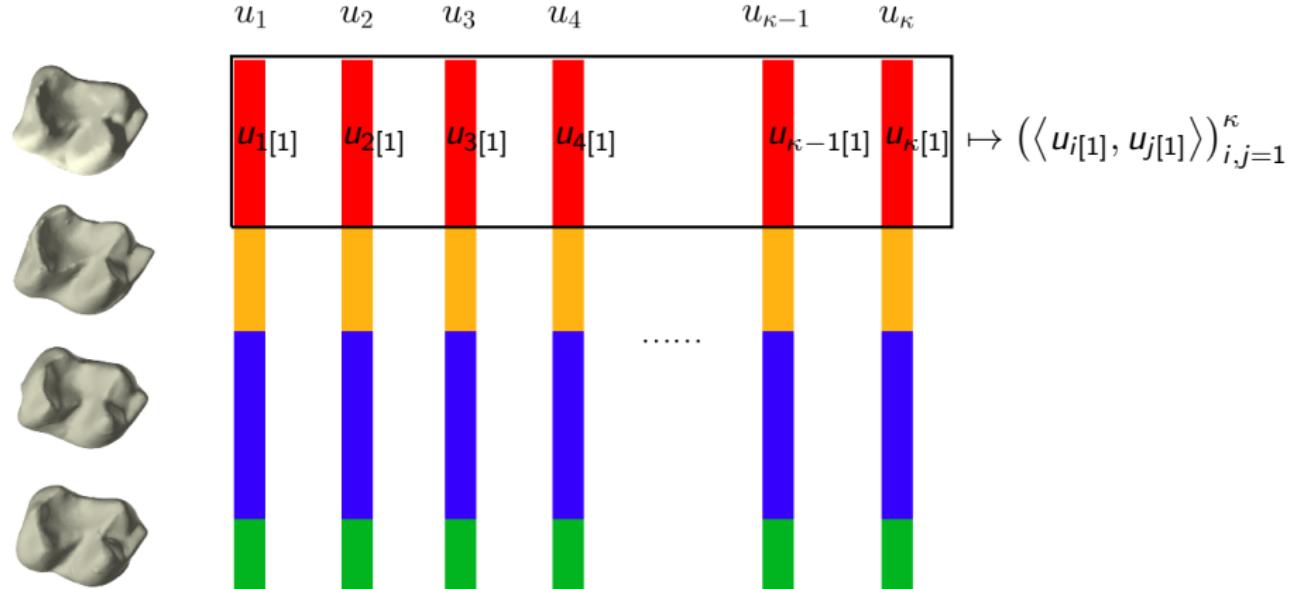
Automatic Landmarking — Interpretability



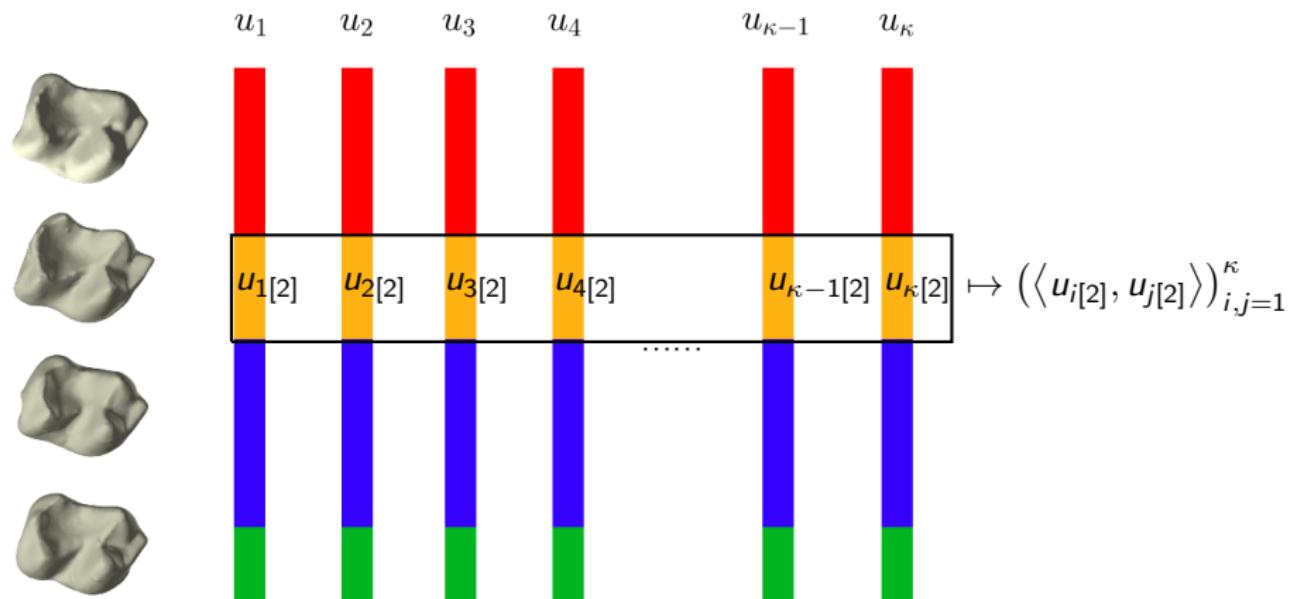
HDM Task 2: Embedding the Base Manifold



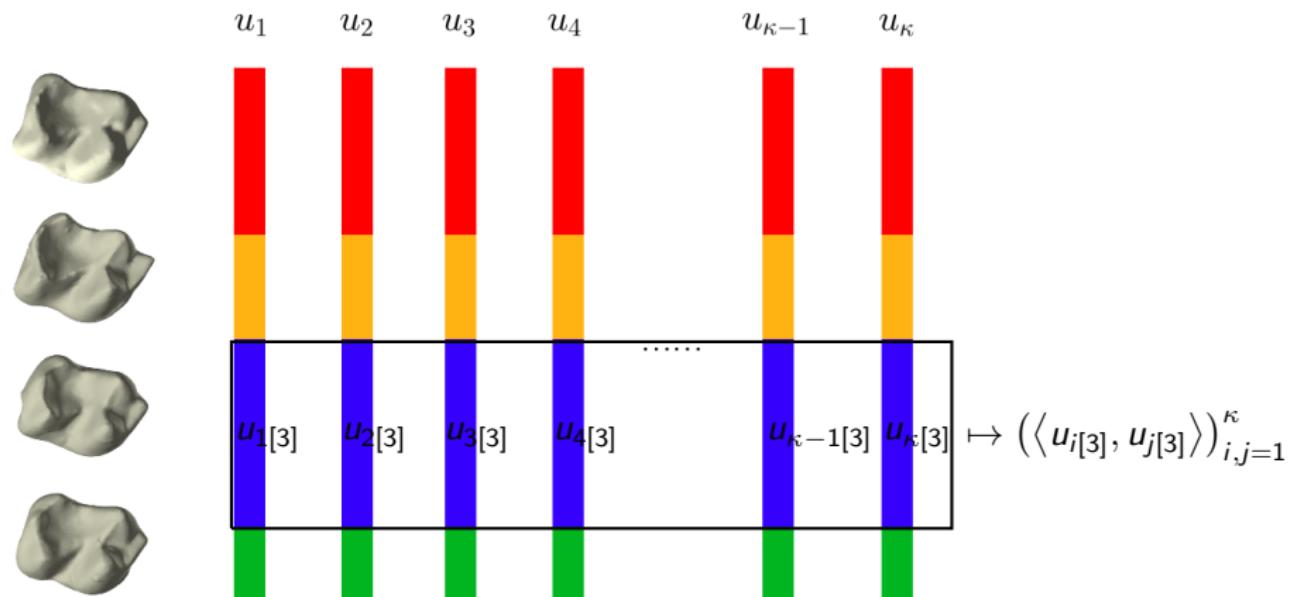
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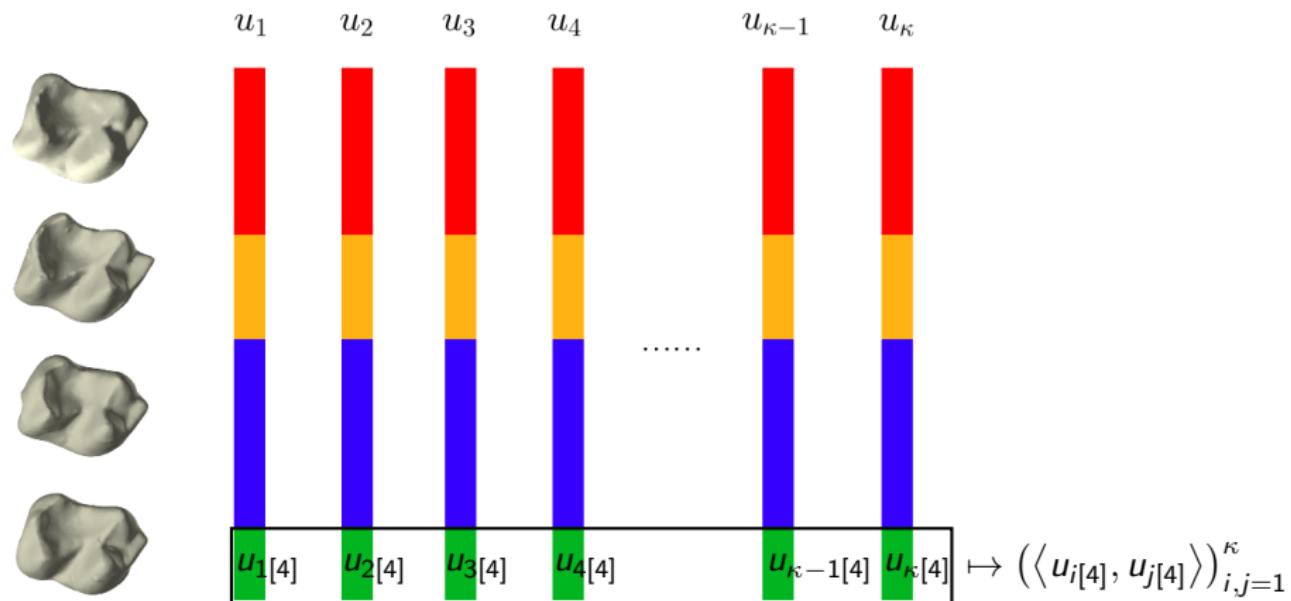
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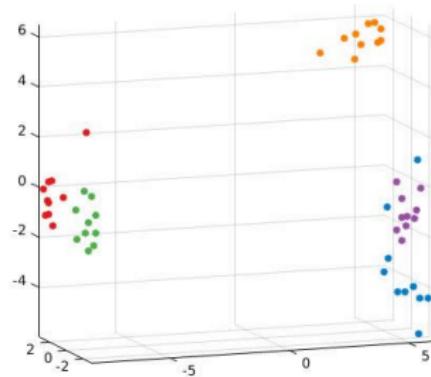
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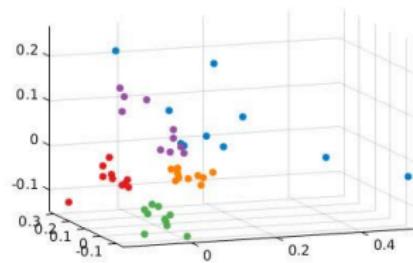
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Species Clustering

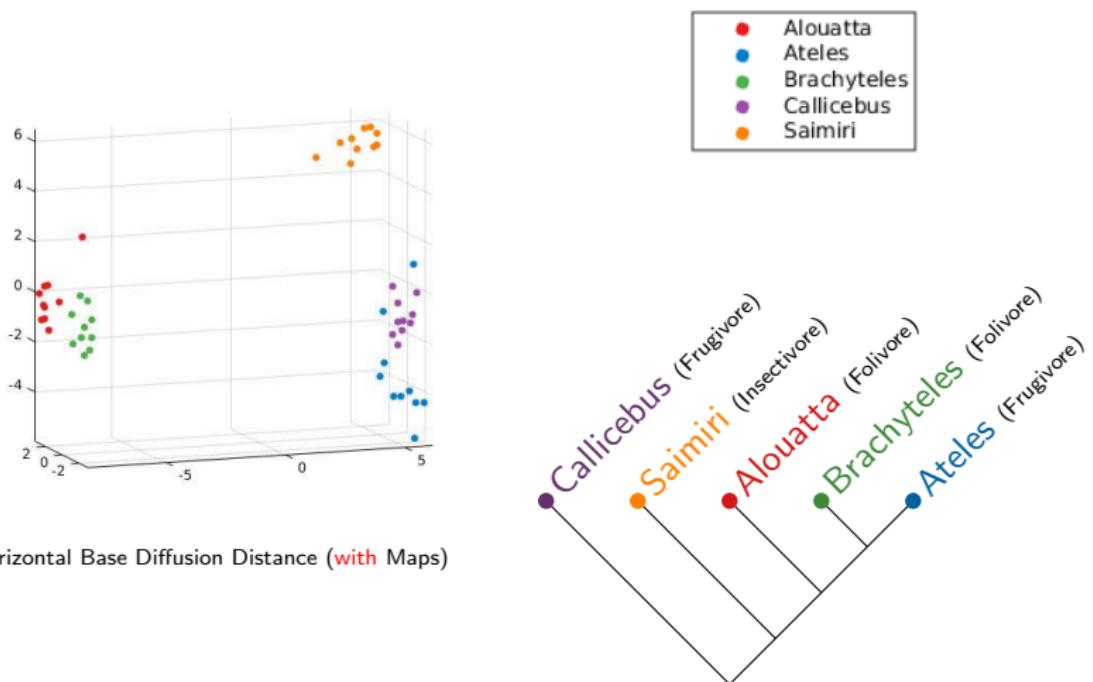


Horizontal Base Diffusion Distance (with Maps)



Diffusion Distance (without Maps)

Species Clustering



HDM: Continuous Limit

M – base manifold; F – template fibre; E – total manifold;
 F_x – fibre over $x \in M$; $P_{yx} : F_x \rightarrow F_y$ – parallel transport

For all $f \in L^1(E)$ and $v \in F_x$, $w \in F_y$, $\mathcal{D}^{-1}\mathcal{W}$ converges to

$$H_{\epsilon,\delta} f(x, v) = \frac{\int_M \int_{F_y} K_{\epsilon,\delta}(x, v; y, w) f(y, w) d\text{vol}_{F_y}(w) d\text{vol}_M(y)}{\int_M \int_{F_y} K_{\epsilon,\delta}(x, v; y, w) d\text{vol}_{F_y}(w) d\text{vol}_M(y)}$$

as $\epsilon, \delta \rightarrow 0$ and $n, \ell_1, \dots, \ell_n \rightarrow \infty$, where

$$K_{\epsilon,\delta}(x, v; y, w) = \exp \left(-\frac{d_M^2(x, y)}{\epsilon} - \frac{d_{F_y}^2(P_{yx}v, w)}{\delta} \right)$$

Asymptotic Theory: Generator of the Diffusion Process

Theorem (G., 2019). If $\delta = O(\epsilon)$ as $\epsilon \rightarrow 0$, then for any $f \in C^\infty(E)$ and $(x, v) \in E$, as $\epsilon \rightarrow 0$,

$$\begin{aligned} & H_{\epsilon, \delta} f(x, v) \\ &= f(x, v) + \epsilon \frac{m_{21}}{2m_0} \Delta_H f(x, v) + \delta \frac{m_{22}}{2m_0} \Delta_E^V f(x, v) + O(\epsilon^2 + \delta^2). \end{aligned}$$

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- ▶ Δ_H is the Bochner horizontal Laplacian on E

Asymptotic Theory: Generator of the Diffusion Process

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- ▶ Δ_E^V is the vertical Laplacian on E
- ▶ Δ_H is the Bochner horizontal Laplacian on E
- ▶ In general $\Delta_H + \Delta_E^V \neq \Delta_E$, true if and only if π is *harmonic*
⇒ HDM is not applying diffusion maps to the total space!

Convergence Rate: Diffusion Maps (DM) vs HDM

Assume M is a Riemannian manifold of dimension d , and denote UTM for the unit tangent bundle of M . Apply DM to M and UTM ; apply HDM to UTM with $\epsilon = \delta$.

	base	fibre	bias	variance
DM (Singer, 2006)	N		$O(\epsilon)$	$O(N^{-1/2} \epsilon^{-d/4})$
DM on UTM	N_B	N_F	$O(\epsilon)$	$O((N_F N_B)^{-1/2} \epsilon^{-d/2+1/4})$
HDM (G., 2019)	N_B	N_F	$O(\epsilon)$	$O(\theta_*^{-1} N_B^{-1/2} \epsilon^{-d/4})$

where

$$\theta_* = 1 - \frac{1}{1 + N_F^{1/2} N_B^{-1/2} \epsilon^{d/2-1/4}}$$

- Amit Singer. "From graph to manifold Laplacian: the convergence rate." *Applied and Computational Harmonic Analysis*, 21 (1), 135–144 (2006)
- Tingran Gao. "The Diffusion Geometry of Fibre Bundles: Horizontal Diffusion Maps." *Applied and Computational Harmonic Analysis*, online first, 1–69 (2019)

Outline

Background

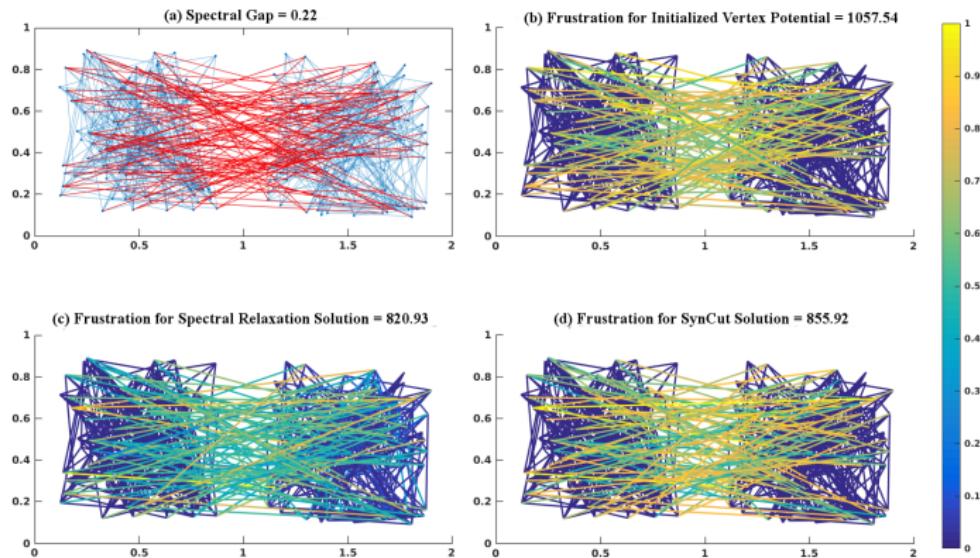
- ▶ Spectral Geometry and Data Analysis

Manifold Learning on Fibre Bundles

- ▶ Motivation: Comparative Biology
- ▶ A Fibre Bundle Vision
- ▶ Horizontal Diffusion Maps

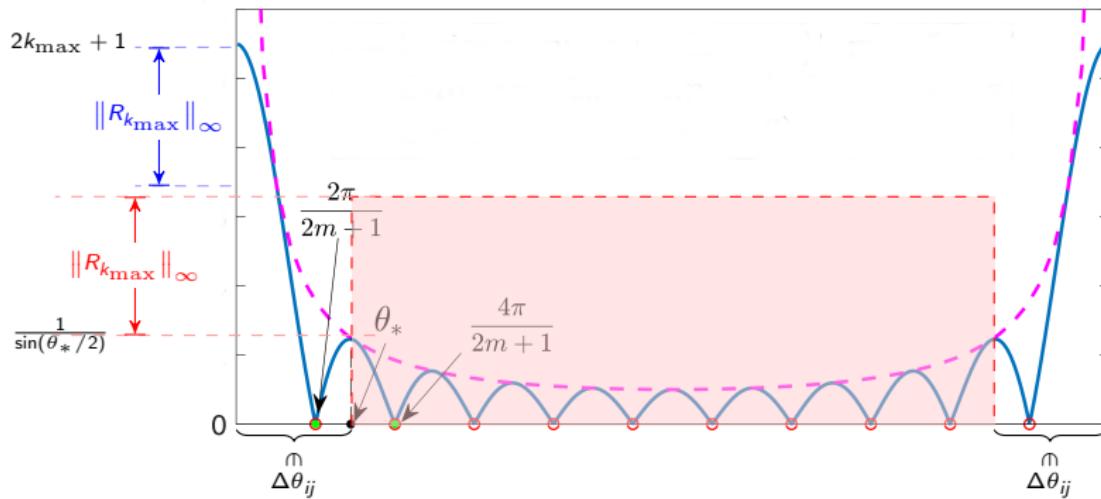
Other Projects

Synchronization and Community Detection



- Tingran Gao, Jacek Brodzki, Sayan Mukherjee. "The Geometry of Synchronization Problems and Learning Group Actions." *Discrete & Computational Geometry*, online first, pp.1–62 (2019)
- Chandrajit Bajaj, Tingran Gao, Zihang He, Qixing Huang, Zhenxiao Liang, "SMAC: Simultaneous Mapping and Clustering Using Spectral Decompositions." *Proceedings of the 35th International Conference on Machine Learning, PMLR* 80:324–333 (2018)

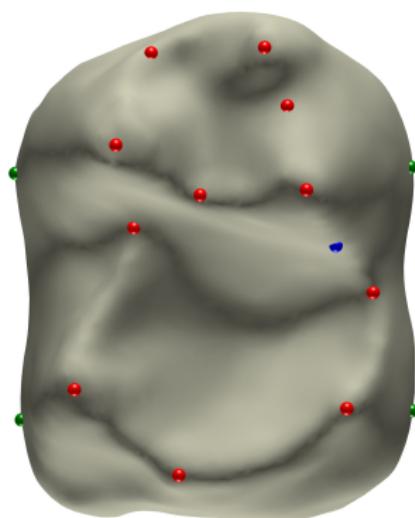
Multi-Representation Manifold Learning on Fibre Bundles



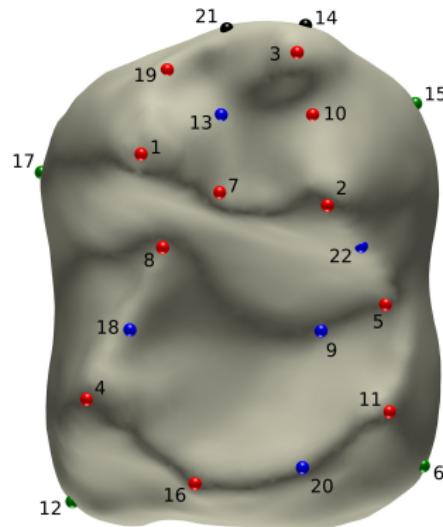
- Tingran Gao, Zhizhen Zhao. "Multi-Frequency Phase Synchronization." *Proceedings of the 36th International Conference on Machine Learning (ICML 2019)*, PMLR 97:2132–2141, 2019.
- Tingran Gao, Yifeng Fan, Zhizhen Zhao. "Representation Theoretic Patterns in Multi-Frequency Class Averaging for Three-Dimensional Cryo-Electron Microscopy." arxiv:1906.01082.
- Yifeng Fan, Tingran Gao, Zhizhen Zhao. "Unsupervised Co-Learning on \mathcal{G} -Manifolds Across Irreducible Representations." *33rd Conference on Neural Information Processing Systems (NeurIPS 2019)*.

Gaussian Process Landmarking

Sequential Experimental Design in Manifold Learning



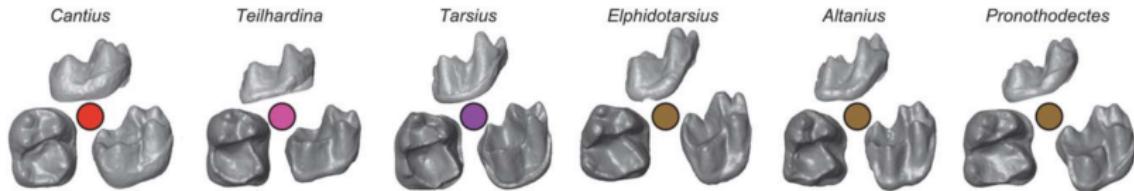
(a) Observer Landmarks



(b) First 22 Gaussian Process Landmarks

- Tingran Gao, Shahar Z. Kovalsky, Ingrid Daubechies. "Gaussian Process Landmarking on Manifolds." *SIAM Journal on Mathematics of Data Science*, 1(1), 208–236 (2019)
- Tingran Gao, Shahar Z. Kovalsky, Doug M. Boyer, Ingrid Daubechies. "Gaussian Process Landmarking for Three-Dimensional Geometric Morphometrics." *SIAM Journal on Mathematics of Data Science*, 1(1), 237–267 (2019)

Computational Geometry in Scientific Applications



- Tingran Gao, Gabriel S. Yapuncich, Ingrid Daubechies, Sayan Mukherjee, Doug M. Boyer. "Development and Assessment of Fully Automated and Globally Transitive Geometric Morphometric Methods, with Application to a Biological Comparative Dataset with High Interspecific Variation." *The Anatomical Record: Advances in Integrative Anatomy and Evolutionary Biology*, 301 (4), 636–658 (2017)
- Natasha S. Vitek, Carly L. Manz, Tingran Gao, Jonathan I. Bloch, Suzanne G. Strait, Doug M. Boyer. "Semi-Supervised Determination of Pseudocryptic Morphotypes Using Observer-Free Characterizations of Anatomical Alignment and Shape." *Ecology and Evolution*, 7(14), 5041–5055 (2017)
- Courtney P. Orsbon, Nicholas J. Gidmark, Tingran Gao, Callum F. Ross. "A Hydraulic Mechanism of Tongue Base Retraction During Swallowing in Primates." submitted (2019)
- Katie Collins, Stewart Edie, Tingran Gao, Rudiger Bieler, David Jablonski. "Spatial Filters of Function and Phylogeny Determine Morphological Disparity with Latitude." *PLoS ONE*, 14(8): e0221490 (2019)
- Bruce Wang, Timothy Sudijono, Henry Kirveslahti, Tingran Gao, Doug M. Boyer, Sayan Mukherjee, Lorin Crawford. "A Statistical Pipeline for Identifying Physical Features that Differentiate Classes of 3D Shapes." under review (2019), bioRxiv doi:10.1101/701391

Collaborators



Doug Boyer
Duke



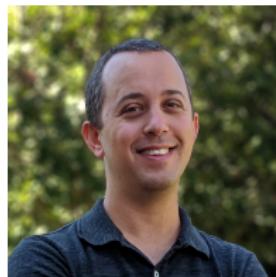
Ingrid Daubechies
Duke



Sayan Mukherjee
Duke



Jacek Brodzki
Southampton



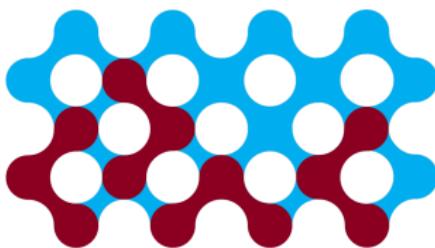
Shahar Kovalsky
Duke



Zhizhen Zhao
UIUC

Thank You!

- ▶ AMS–Simons Travel Grant
- ▶ UChicago CDAC Data Science Discovery Seed Grant
- ▶ NSF CDS&E-MSS DMS-1854831



Center for Data and Computing

- Tingran Gao. "The Diffusion Geometry of Fibre Bundles: Horizontal Diffusion Maps." *Applied and Computational Harmonic Analysis*, online first, pp.1–69 (2019)
- Tingran Gao, Jacek Brodzki, Sayan Mukherjee. "The Geometry of Synchronization Problems and Learning Group Actions," *Discrete & Computational Geometry*, online first, pp.1–62 (2019)